Compatibility Conditions for Singular Strain Fields on Linear Shells

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Abstract

Strain compatibility conditions are derived for orientable linear shells with strain fields which are piecewise smooth and develop concentrations along curves and junctions on the shell surface. The concentrations in strain fields appear due to folds, fold dipoles, and tears in the shell. The compatibility equations include local conditions in the bulk of the shell surface, local equations at the singular curves in terms of jump in strain fields (and their gradients) and strain concentrations, local equations at the junction of singular curves, and global conditions for every family of irreducible curves on the shell surface. These equations couple shell kinematics with the geometry and topology of the shell surface. The applicability of the framework is illustrated assuming the bulk strains to be elastic and the strain concentrations to be plastic. The compatibility equations then lead to a boundary value problem for the determination of stresses and moments in an elastic surface in response to the incompatibility induced by the plastic strains.

Keywords: Strain compatibility; Linear shells; Singular strain fields; Strain concentrations; Folds on shells; Topological restrictions; plastic folds and tears

1 Introduction

The strain compatibility conditions for *smooth* strain fields on linearised elastic shells are well known in the shell literature [4]. These conditions when stated in terms of the stress functions, derivable from the equilibrium equations, yield a boundary value problem for the determination of stress functions [5]. Such stress based formulations are essential when the sources of stress are internal and are given in terms of strain incompatibility, e.g., in the problems related to thermoelasticity, biological growth, and micromechanics of defects [1,13,15,16]. In several of the problems, however, the strain fields are necessarily singular. This may be due to the nature of material/geometric constraints or because of the intrinsic

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character of imposed incompatibility [7, 10, 14]. The purpose of the present article is to derive the strain compatibility equations for *singular* strain fields on shell surfaces with linearised kinematics.

As one of our main results we establish the strain compatibility conditions for shells in a distributional sense (Lemma 3.1). This allows us to obtain the localised compatibility conditions for strains fields which demonstrate singular behaviour in certain parts of the domain. A similar methodology was introduced by the authors in the context of linear elasticity and von Kármán plate theory [8–10]. The mathematical apparatus built in our previous works, however, is not applicable directly to the problem at hand. For shells, most importantly, we need to introduce distributions on curved surfaces and be careful about both the local geometry of the shell surface and its topological attributes. The linear shell is assumed to carry a stretching strain and a bending strain distribution. In the present work, singularities in strain fields appear either due to them being discontinuous across interfacial curves on the surface or due to strain concentrations defined over interfacial curves and junction points. Discontinuities in strain fields can appear when a subdomain of the shell, separated with the rest of the surface through an interface, is exposed to stretching and bending eigenstrains, as in the problems of phase transformation and inhomogeneities. On the other hand, strain concentrations along interfacial curves can appear due to *tearing* (displacement discontinuity) or *folding* (slope discontinuity) in the deformed surface. In particular, an interfacial concentration in stretching strain appears due to in-plane tangential jumps in the displacement field. A monopolar interfacial concentration of bending strain develops in response to displacement and slope discontinuities whereas a dipolar interfacial concentration of bending strain appears due to normal jumps in the displacement field. It is interesting to note that, when multiple tears (along various interfacial curves) meet at a junction within the shell domain, a point concentration of bending strain necessarily appears at the junction point.

The distributional strain compatibility conditions, after assuming a certain form for distributional strains, are shown to be equivalent to a set of local compatibility conditions in the bulk of shell surface (away from singular domains), at the singular curves, and at the junction point, and a global compatibility condition for each independent family of irreducible curves on the shell. The local conditions on the singular domain are given in terms of jump in bulk strain fields (and their gradients) and strain concentrations. All the conditions, except for those in the bulk, are novel. In particular, our work provides a rigorous framework to discuss problems associated with compatible/incompatible strains arising out of folds and tears, along lines of arbitrary curvature, on curved geometries with different topologies, albeit for linearised kinematics. As an illustrative example, we consider folds and tears to be describable in terms of plastic strain concentrations and consider the strains in the bulk (away from singularities) to be necessarily elastic. The elastic strains in the bulk, and hence the associated stress and moment fields, appear as a result of an incompatibility problem with sources of incompatibility arising out of the plastic strain concentrations.

The paper is organised as follows. In Section 2 we provide the necessary mathematical background for our work. We introduce the idea of distributions on curved geometries and derive several results associated with surface derivatives and surface gradients of distributions. We discuss the existence and regularity of a scalar potential such that a given vector potential can be expressed as a gradient of the former, all in the sense of distributions over curved geometries and arbitrary topologies. This would become the backbone of the rest of our work in subsequent sections. In Section 3.1, local strain-displacement relations are derived assuming the displacement field to be piecewise smooth (possibly discontinuous). The emergence of concentrations in strain fields, in response to the discontinuities in displacement (and its gradient), is emphasised. The strain compatibility results are discussed in Section 3.2. For a given prescription of strain fields, in terms of piecewise smooth bulk fields and interfacial (and junction) concentrations, we seek to derive the necessary and sufficient conditions for there to exist a displacement field which satisfies the strain-displacement relations. Finally, in Section 4 we pose a boundary value problem for the determination of stresses and moments in an elastic surface in response to the incompatibility induced by strain concentrations.

2 Mathematical preliminaries

2.1 Notation

We use $\mathbb{R}^3$ to denote the 3D (three-dimensional) Euclidean space (and also the 3D Euclidean vector space) and Lin as the space of second order tensors on $\mathbb{R}^3$. Given two vectors $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R}^3$, $\boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \in$ Lin is the tensor product and $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle \in \mathbb{R}$ is the inner product. The inner product for two m-order tensors is defined such that, for $\boldsymbol{a}_i, \boldsymbol{b}_i \in \mathbb{R}^3$, $i = 1, \dots, m$, $\langle \boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \cdots \otimes \boldsymbol{a}_m, \boldsymbol{b}_1 \otimes \boldsymbol{b}_2 \cdots \otimes \boldsymbol{b}_m \rangle = \Pi_{i=1}^m \langle \boldsymbol{a}_i, \boldsymbol{b}_i \rangle$, where $\Pi_{i=1}^m$ indicates the product of m terms. For a n-order tensor $\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \cdots \otimes \boldsymbol{a}_n$ and a m-order tensor $\boldsymbol{b}_1 \otimes \boldsymbol{b}_2 \cdots \otimes \boldsymbol{b}_m$, with $n > m$, $(\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \cdots \otimes \boldsymbol{a}_n)(\boldsymbol{b}_1 \otimes \boldsymbol{b}_2 \cdots \otimes \boldsymbol{b}_m) = (\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \cdots \otimes \boldsymbol{a}_{n-m}) \Pi_{i=1}^m \langle \boldsymbol{a}_{n-m+i}, \boldsymbol{b}_i \rangle$. For instance, given third order tensors $\boldsymbol{A}$ and $\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \boldsymbol{a}_3$, we have $\langle \boldsymbol{A}, \boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \boldsymbol{a}_3 \rangle = \langle \boldsymbol{a}_1, \boldsymbol{A}(\boldsymbol{a}_2 \otimes \boldsymbol{a}_3) \rangle$. A cross product between a vector $\boldsymbol{v} \in \mathbb{R}^3$ and a second order tensor $\boldsymbol{a}_1 \otimes \boldsymbol{a}_2$ is a second order tensor such that $\boldsymbol{v} \times (\boldsymbol{a}_1 \otimes \boldsymbol{a}_2) = (\boldsymbol{v} \times \boldsymbol{a}_1) \otimes \boldsymbol{a}_2$ and $(\boldsymbol{a}_1 \otimes \boldsymbol{a}_2) \times \boldsymbol{v} = \boldsymbol{a}_1 \otimes (\boldsymbol{a}_2 \times \boldsymbol{v})$. The subspace of symmetric second order tensors is denoted as Sym. For any $\boldsymbol{B} \in$ Lin, $\text{sym}(\boldsymbol{B}) \in$ Sym is the symmetric part of $\boldsymbol{B}$.

Let $S \subset \mathbb{R}^3$ be an orientable, regular, and bounded surface embedded in 3D Euclidean space. The unit normal field to S is given by a smooth function $\boldsymbol{n} : S \to \mathbb{R}^3$ with $\langle \boldsymbol{n}, \boldsymbol{n} \rangle = 1$. Let $\mathbb{P} = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}$ be the projection tensor, where $\boldsymbol{I} \in$ Lin is the identity tensor. Given differentiable functions $f : S \to \mathbb{R}$ and $\boldsymbol{u} : S \to \mathbb{R}^3$, we denote their surface gradients as, respectively, $\nabla_S f : S \to \mathbb{R}^3$, a tangential vector field (i.e., $\langle \nabla_S f, \boldsymbol{n} \rangle = 0$), and $\nabla_S \boldsymbol{u} : S \to$ Lin, a tangential tensor field (i.e., $(\nabla_S \boldsymbol{u})\boldsymbol{n} = \boldsymbol{0}$). Given differentiable functions $\boldsymbol{u} : S \to \mathbb{R}^3$, $\boldsymbol{v} : S \to$ Lin, we denote their surface divergence as, respectively, $\text{div}_S\, \boldsymbol{u} : S \to \mathbb{R}$ and $\text{div}_S\, \boldsymbol{v} : S \to \mathbb{R}^3$. The metric of the surface S is given by $\mathbb{P}$. On the other hand, $\boldsymbol{b} : S \to$ Sym is a surface tensor (i.e., $\boldsymbol{b}\boldsymbol{n} = \boldsymbol{b}^T \boldsymbol{n} = \boldsymbol{0}$, where T denotes the transpose) such that $\boldsymbol{b} = -\nabla_S \boldsymbol{n}$; $\boldsymbol{b}$ represents the second fundamental form of S. The scalar field $k = -\,\text{div}_S\, \boldsymbol{n}$ is twice the mean curvature of S. Given a smooth vector field $\boldsymbol{u} : S \to \mathbb{R}^3$, the surface divergence theorem takes the following form:

$$\int_S \text{div}_S\, \boldsymbol{u}\, \text{da} = \int_{\partial S} \langle \boldsymbol{u}, \boldsymbol{\nu} \rangle\, \text{dl} - \int_S k \langle \boldsymbol{u}, \boldsymbol{n} \rangle\, \text{da}, \tag{1}$$

where ∂S is the boundary of S, $\boldsymbol{\nu}$ is the in-plane unit normal vector to ∂S, pointing in the outward direction (away from S), dl is the length measure on ∂S, and da is the area measure on S.

Let $C \subset S$ be a regular, bounded curve embedded on the surface S. Let $\{\boldsymbol{t}, \boldsymbol{\nu}, \boldsymbol{n}\}$ be the triad representing the Darboux frame of the curve C, where $\boldsymbol{t}$ is the unit tangent vector to the curve C and $\boldsymbol{\nu}$ is the in-plane unit normal vector to C such that $\boldsymbol{t} \times \boldsymbol{\nu} = \boldsymbol{n}$. The geodesic curvature k_g, normal curvature k_n, and geodesic torsion τ_g are defined by

$$k_g = \left\langle \frac{d\boldsymbol{t}}{ds}, \boldsymbol{\nu} \right\rangle, \; k_n = \left\langle \frac{d\boldsymbol{t}}{ds}, \boldsymbol{n} \right\rangle, \text{ and } \tau_g = \left\langle \frac{d\boldsymbol{\nu}}{ds}, \boldsymbol{n} \right\rangle, \tag{2}$$

respectively, where s is the arc length parameter on C. In terms of these curvatures, the second fundamental form at a point on C can be written as

$$\boldsymbol{b} = k_n \boldsymbol{t} \otimes \boldsymbol{t} + \tau_g (\boldsymbol{t} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{t}) + (k - k_n) \boldsymbol{\nu} \otimes \boldsymbol{\nu}. \tag{3}$$

2.2 Distributions on curved surfaces

Let $\mathbb{V}$ be a Euclidean vector space, $\mathcal{D}(S, \mathbb{V})$ be the space of compactly supported $\mathbb{V}$ valued functions on S, and $\mathcal{D}'(S, \mathbb{V})$ be the dual to $\mathcal{D}(S, \mathbb{V})$; $\mathcal{D}'(S, \mathbb{V})$ is the space of $\mathbb{V}$ valued distributions on S [3]. The spaces $\mathcal{D}(S, \mathbb{R})$ and $\mathcal{D}'(S, \mathbb{R})$ are written as $\mathcal{D}(S)$ and $\mathcal{D}'(S)$, respectively. Given an integrable function $\boldsymbol{f} : S \to \mathbb{V}$, we define an associated distribution $\boldsymbol{T}_f \in \mathcal{D}'(S, \mathbb{V})$ such that $\boldsymbol{T}_f(\boldsymbol{\phi}) = \int_S \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle \, \mathrm{da}$, for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathbb{V})$. The product of two smooth functions can be generalised to the product of a smooth function $f : S \to \mathbb{R}$ with a distribution $T \in \mathcal{D}'(S)$ as $fT(\phi) = T(f\phi)$, for all $\phi \in \mathcal{D}(S)$.

We say that a distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ is *tangential* if $\boldsymbol{T}(\phi \boldsymbol{n}) = 0$ for all $\phi \in \mathcal{D}(S, \mathbb{R}^3)$. A distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ is *normal* to the surface S if $\boldsymbol{T}(\boldsymbol{\phi}) = 0$ for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathbb{R}^3)$ such that $\langle \boldsymbol{\phi}, \boldsymbol{n} \rangle = 0$. It follows immediately from these definitions that any distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ can be decomposed as $\boldsymbol{T} = \mathbb{P}\boldsymbol{T} + (\boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{T}$ into its tangential and normal components.

2.3 Surface derivatives of distributions

Given a distribution $T \in \mathcal{D}'(S)$, we define its distributional surface gradient $\nabla_S T \in \mathcal{D}'(S, \mathbb{R}^3)$ as

$$\nabla_S T(\boldsymbol{\phi}) = -T(\mathrm{div}_S(\mathbb{P}\boldsymbol{\phi})), \tag{4}$$

for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathbb{R}^3)$. This is equivalent to writing $\nabla_S T(\boldsymbol{\phi}) = -T(\mathrm{div}_S \, \boldsymbol{\phi} + k \langle \boldsymbol{\phi}, \boldsymbol{n} \rangle)$. The distributional surface gradient generalises the notion of surface gradient of a differentiable surface function, i.e., $T_{\nabla_S f} = \nabla_S (T_f)$ for any differentiable function $f : S \to \mathbb{R}$. Analogously, the surface gradient of a vector valued distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{V})$ is a distribution $\nabla_S \boldsymbol{T} \in \mathcal{D}'(S, \mathbb{V} \otimes \mathbb{R}^3)$ such that

$$\nabla_S \boldsymbol{T}(\boldsymbol{\phi}) = -\boldsymbol{T}(\mathrm{div}_S(\boldsymbol{\phi}\mathbb{P})), \tag{5}$$

for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathrm{Lin})$.

The surface divergence of a vector valued distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{V})$ is a scalar distribution $\mathrm{Div}_S \, \boldsymbol{T} \in \mathcal{D}'(\Omega)$ defined by

$$\mathrm{Div}_S \, \boldsymbol{T}(\phi) = -\boldsymbol{T}(\nabla_S \phi - k\phi \boldsymbol{n}), \tag{6}$$

for all $\phi \in \mathcal{D}(S)$. The surface divergence of a tensor valued distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathrm{Lin})$ is a vector valued distribution $\mathrm{Div}_S\, \boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ defined by

$$\mathrm{Div}_S\, \boldsymbol{T}(\boldsymbol{\phi}) = -\boldsymbol{T}(\nabla_S \boldsymbol{\phi} - k\boldsymbol{\phi} \otimes \boldsymbol{n}), \tag{7}$$

for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathbb{R}^3)$.

2.4 Identities for distributional derivatives

We will be working with the following types of distributions on S:

(i) $\mathcal{B}(S) \subset \mathcal{D}'(S)$, such that for every distribution $B \in \mathcal{B}(S)$ we can write $B(\phi) = \int_S b\phi\, \mathrm{da}$, for all $\phi \in \mathcal{D}(S)$, where $b : S \to \mathbb{R}$ is a piecewise smooth and uniformly bounded function which can be discontinuous across curve $C \subset S$.

(ii) $\mathcal{I}(S) \subset \mathcal{D}'(S)$, such that for every distribution $I \in \mathcal{I}(S)$ we can write $I(\phi) = \int_C c\phi\, \mathrm{dl}$, for all $\phi \in \mathcal{D}(S)$, where $c : C \to \mathbb{R}$ is a smooth and uniformly bounded function on curve $C \subset S$ and dl is the length measure on C.

(iii) $\mathcal{F}(S) \subset \mathcal{D}'(S)$, such that for every distribution $F \in \mathcal{F}(S)$ we can write $F(\phi) = \int_C f(\partial\phi/\partial\nu)\, \mathrm{dl}$, for all $\phi \in \mathcal{D}(S)$, where $f : C \to \mathbb{R}$ is a smooth and uniformly bounded function on curve $C \subset S$ and $\partial/\partial\nu$ is the partial derivative along $\boldsymbol{\nu}$.

Similar definitions can be introduced for analogous distributions which are vector or tensor valued. While the distribution $\mathcal{B}(S)$ represents a piecewise smooth scalar field on S, the distributions $\mathcal{I}(S)$ and $\mathcal{F}(S)$ represent a smooth concentration field and a smooth concentration dipole field on $C \subset S$, respectively. A combination of these distributions can be used to represent piecewise smooth fields on S which may concentrate on C. In the following, we collect identities dealing with surface gradients and surface divergences of the kind of distributions defined above. These formulae are useful in obtaining local point-wise implications of mathematical statements made in terms of distributions.

Identities 2.1. *(Surface gradient of distributions) For all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$,*
(a) If $B \in \mathcal{B}(S)$, such that $B(\phi) = \int_S b\phi\, \mathrm{da}$, for all $\phi \in \mathcal{D}(S)$, where b is a piecewise smooth and uniformly bounded scalar field on S, then

$$\nabla_S B(\boldsymbol{\psi}) = \int_S \langle \nabla_S b, \boldsymbol{\psi} \rangle\, \mathrm{da} - \int_C [\![b]\!] \langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle\, \mathrm{dl}, \tag{8}$$

where $[\![b]\!] : C \to \mathbb{R}$ represents the jump in b across C, i.e., $[\![b]\!](\boldsymbol{x}) = b^+(\boldsymbol{x}) - b^-(\boldsymbol{x})$ with $b^{\pm}(\boldsymbol{x}) = \lim_{p \to 0, p \in \mathbb{R}^+} b(\boldsymbol{x} \mp p\boldsymbol{\nu})$ for $\boldsymbol{x} \in C$.
(b) If $I \in \mathcal{I}(S)$, such that $I(\phi) = \int_C c\phi\, \mathrm{dl}$, for all $\phi \in \mathcal{D}(S)$, where c is a smooth and uniformly bounded scalar field on curve $C \subset S$, then

$$\nabla_S I(\boldsymbol{\psi}) = \int_C \left\langle \left(\frac{dc}{ds}\boldsymbol{t} + k_g c\boldsymbol{\nu} - c(k - k_n)\boldsymbol{n} \right), \boldsymbol{\psi} \right\rangle \mathrm{dl} - \int_C \left\langle c\boldsymbol{\nu}, \frac{\partial \boldsymbol{\psi}}{\partial \nu} \right\rangle \mathrm{dl} - \langle c\boldsymbol{t}, \boldsymbol{\psi} \rangle \,|_{\partial C - \partial S}, \tag{9}$$

where $\partial C - \partial S$ is the set of boundary points of C (if any) which do not intersect with boundary points of S (if any). The notation $|_{\partial C-\partial S}$ indicates that the term is to be evaluated at such boundary points.
(c) If $F \in \mathcal{F}(S)$, such that $F(\phi) = \int_C f(\partial\phi/\partial\nu)\,\mathrm{dl}$, for all $\phi \in \mathcal{D}(S)$, where f is a smooth and uniformly bounded scalar field on curve $C \subset S$, then

$$\nabla_S F(\boldsymbol{\psi}) = \int_C \Bigg\langle \left(f\frac{dk_g}{ds} + k_g\frac{df}{ds} + (k-k_n)\tau_g f \right)\boldsymbol{t} + \left(f{k_g}^2 - f{\tau_g}^2 + k(k-k_n)f \right)\boldsymbol{\nu}, \boldsymbol{\psi} \Bigg\rangle + \Bigg\langle \left(\tau_g\frac{df}{ds} + (k-k_n)k_g f + f\left\langle \nabla_S\nabla_S\boldsymbol{n}, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\otimes\boldsymbol{\nu}\right\rangle \right)\boldsymbol{n}, \boldsymbol{\psi} \Bigg\rangle + \Bigg\langle \left(\frac{df}{ds}\boldsymbol{t} + k_g f\boldsymbol{\nu} - 2(k-k_n)f\boldsymbol{n} \right), \frac{\partial\boldsymbol{\psi}}{\partial\nu} \Bigg\rangle\,\mathrm{dl} - \int_C \langle f\boldsymbol{\nu}, (\nabla_S\nabla_S\boldsymbol{\psi})\boldsymbol{\nu}\otimes\boldsymbol{\nu}\rangle\,\mathrm{dl} - \left(\langle (k_g f\boldsymbol{t} + \tau_g f\boldsymbol{n}), \boldsymbol{\psi}\rangle + \left\langle f\boldsymbol{t}, \frac{\partial\boldsymbol{\psi}}{\partial\nu} \right\rangle \right)\Big|_{\partial C-\partial S}. \tag{10}$$

Proof. For $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$, let $\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \psi_3\boldsymbol{n}$ where $\boldsymbol{\psi}_0 = \mathbb{P}\boldsymbol{\psi}$ and $\psi_3 = \langle\boldsymbol{\psi}, \boldsymbol{n}\rangle$.
(a) We have $\nabla_S B(\boldsymbol{\psi}) = -B(\mathrm{div}_S\,\boldsymbol{\psi}_0) = -\int_S b\,\mathrm{div}_S\,\boldsymbol{\psi}_0\,\mathrm{da}$, which can be rewritten using the chain rule and the divergence theorem as $\int_S \langle\nabla_S b, \boldsymbol{\psi}\rangle\,\mathrm{da} - \int_C \langle [\![b]\!]\boldsymbol{\nu}, \boldsymbol{\psi}\rangle\,\mathrm{dl}$.
(b) We have $\nabla_S I(\boldsymbol{\psi}) = -I(\mathrm{div}_S\,\boldsymbol{\psi}_0) = -\int_C c\left(\left\langle \frac{\partial\boldsymbol{\psi}_0}{\partial s}, \boldsymbol{t}\right\rangle + \left\langle \frac{\partial\boldsymbol{\psi}_0}{\partial\nu}, \boldsymbol{\nu}\right\rangle \right)\mathrm{dl}$. Thereupon we use $\boldsymbol{\psi}_0 = \boldsymbol{\psi} - \psi_3\boldsymbol{n}$ and $\left\langle \frac{\partial\boldsymbol{n}}{\partial\nu}, \boldsymbol{\nu}\right\rangle = (k_n - k)$, in addition to the chain rule, to obtain the desired result.
(c) We have $\nabla_S F(\boldsymbol{\psi}) = -F(\mathrm{div}_S\,\boldsymbol{\psi}_0) = -\int_C \left(f\frac{\partial}{\partial\nu}(\mathrm{div}_S\,\boldsymbol{\psi}_0)\right)\mathrm{dl}$. Upon noting that

$$\frac{\partial}{\partial\nu}(\mathrm{div}_S\,\boldsymbol{\psi}_0) = \langle \nabla_S\nabla_S\boldsymbol{\psi}_0, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\otimes\boldsymbol{\nu}\rangle + \langle \nabla_S\nabla_S\boldsymbol{\psi}_0, \boldsymbol{t}\otimes\boldsymbol{t}\otimes\boldsymbol{\nu}\rangle + \langle \nabla_S\nabla_S\boldsymbol{\psi}_0, \boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{\nu}\rangle\,,$$

and using the identities

$$(\nabla_S\nabla_S\boldsymbol{\psi})\boldsymbol{t}\otimes\boldsymbol{\nu} = \frac{\partial}{\partial s}(\nabla_S\boldsymbol{\psi}\boldsymbol{\nu}) + k_g\frac{\partial\boldsymbol{\psi}}{\partial s} \text{ and}$$
$$(\nabla_S\nabla_S\boldsymbol{\psi})\boldsymbol{n}\otimes\boldsymbol{\nu} = \tau_g\frac{\partial\boldsymbol{\psi}}{\partial s} + (k-k_n)\frac{\partial\boldsymbol{\psi}}{\partial\nu}$$

in the following calculations

$$\begin{aligned}
-\int_C f\langle \nabla_S\nabla_S\boldsymbol{\psi}_0, \boldsymbol{t}\otimes\boldsymbol{t}\otimes\boldsymbol{\nu}\rangle\,\mathrm{dl} &= -\int_C \langle f\boldsymbol{t}, (\nabla_S\nabla_S\boldsymbol{\psi}_0)\boldsymbol{t}\otimes\boldsymbol{\nu}\rangle\,\mathrm{dl} = -\int_C \left\langle f\boldsymbol{t}, \left(\frac{d}{ds}(\nabla_S\boldsymbol{\psi}_0\boldsymbol{\nu}) + k_g\frac{\partial\boldsymbol{\psi}_0}{\partial s} \right) \right\rangle\mathrm{dl} \\
&= \int_C \left(\left\langle \frac{d}{ds}(f\boldsymbol{t}), \frac{\partial\boldsymbol{\psi}_0}{\partial\nu} \right\rangle + \left\langle \frac{d}{ds}(k_g f\boldsymbol{t}), \boldsymbol{\psi}_0 \right\rangle \right)\mathrm{dl} - \left(\langle (k_g f\boldsymbol{t}, \boldsymbol{\psi}_0\rangle + \left\langle f\boldsymbol{t}, \frac{\partial\boldsymbol{\psi}_0}{\partial\nu} \right\rangle \right)\Big|_{\partial C-\partial S} \\
&= \int_C \Bigg(\Bigg\langle \left(\left(fk_n\tau_g + \frac{dk_g}{ds}f + k_g\frac{df}{ds} \right)\boldsymbol{t} + \left(fk_n(k-k_n) + {k_g}^2 f \right)\boldsymbol{\nu} + \left(k_g(k-k_n)f + \tau_g\frac{df}{ds} \right)\boldsymbol{n} \right), \boldsymbol{\psi} \Bigg\rangle \\
&\quad + \left\langle \left(\frac{df}{ds}\boldsymbol{t} + fk_g\boldsymbol{\nu} \right), \frac{\partial\boldsymbol{\psi}}{\partial\nu} \right\rangle \Bigg)\mathrm{dl} - \left(\langle (k_g f\boldsymbol{t}, \boldsymbol{\psi}\rangle + \langle (\tau_g f\boldsymbol{n}, \boldsymbol{\psi}\rangle + \left\langle f\boldsymbol{t}, \frac{\partial\boldsymbol{\psi}}{\partial\nu} \right\rangle \right)\Big|_{\partial C-\partial S},
\end{aligned}$$

$$\begin{aligned}
-\int_C f\langle \nabla_S\nabla_S\boldsymbol{\psi}_0, \boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{\nu}\rangle\,\mathrm{dl} &= -\int_C \langle f\boldsymbol{n}, (\nabla_S\nabla_S\boldsymbol{\psi}_0)\boldsymbol{n}\otimes\boldsymbol{\nu}\rangle\,\mathrm{dl} = -\int_C \left\langle f\boldsymbol{n}, \left(\tau_g\frac{\partial\boldsymbol{\psi}_0}{\partial s} + (k-k_n)\frac{\partial\boldsymbol{\psi}_0}{\partial\nu} \right) \right\rangle\mathrm{dl} \\
&= \int_C \left\langle \frac{d}{ds}(\tau_g f\boldsymbol{n}), \boldsymbol{\psi}_0 \right\rangle - \left\langle (k-k_n)f\boldsymbol{n}, \frac{\partial\boldsymbol{\psi}_0}{\partial\nu} \right\rangle\mathrm{dl} = \int_C \left\langle \left(-k\tau_g f\boldsymbol{t} - ((k-k_n)^2 + {\tau_g}^2)f\boldsymbol{\nu} \right), \boldsymbol{\psi} \right\rangle\mathrm{dl}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
-\int_C f\langle \nabla_S\nabla_S\boldsymbol{\psi}_0, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\otimes\boldsymbol{\nu}\rangle\,\mathrm{dl} &= \int_C \Big(\langle -f\boldsymbol{\nu}, (\nabla_S\nabla_S\boldsymbol{\psi})\boldsymbol{\nu}\otimes\boldsymbol{\nu}\rangle + \langle \nabla_S\nabla_S\boldsymbol{n}, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\otimes\boldsymbol{\nu}\rangle\langle f\boldsymbol{n}, \boldsymbol{\psi}\rangle \\
&\quad + \left\langle \left(2(k-k_n)\tau_g f\boldsymbol{t} + 2(k-k_n)^2 f\boldsymbol{\nu} \right), \boldsymbol{\psi} \right\rangle - \left\langle 2(k-k_n)f\boldsymbol{n}, \frac{\partial\boldsymbol{\psi}}{\partial\nu} \right\rangle \Big)\mathrm{dl},
\end{aligned}$$

we can obtain the desired result. $\square$

Identities 2.2. *(Surface divergence of distributions) For all $\psi \in \mathcal{D}(S)$,*
(a) If $\boldsymbol{B} \in \mathcal{B}(S, \mathbb{R}^3)$, such that $\boldsymbol{B}(\boldsymbol{\phi}) = \int_S \langle \boldsymbol{b}, \boldsymbol{\phi} \rangle \,\mathrm{da}$, for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathbb{R}^3)$, where $\boldsymbol{b}$ is a piecewise smooth and uniformly bounded vector field on S, then

$$\mathrm{Div}_S\, \boldsymbol{B}(\psi) = \int_S (\mathrm{div}_S\, \boldsymbol{b}) \psi \,\mathrm{da} - \int_C \langle [\![\boldsymbol{b}]\!], \boldsymbol{\nu} \rangle \psi \,\mathrm{dl}\,. \tag{11}$$

(b) If $\boldsymbol{I} \in \mathcal{I}(S, \mathbb{R}^3)$, such that $\boldsymbol{I}(\boldsymbol{\phi}) = \int_C \langle \boldsymbol{c}, \boldsymbol{\phi} \rangle \,\mathrm{dl}$, for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathbb{R}^3)$, where $\boldsymbol{c}$ is a smooth and uniformly bounded vector field on C, then

$$\mathrm{Div}_S\, \boldsymbol{I}(\psi) = \int_C \left(\left\langle \frac{d\boldsymbol{c}}{ds}, \boldsymbol{t} \right\rangle + k_g \langle \boldsymbol{c}, \boldsymbol{\nu} \rangle - (k - k_n) \langle \boldsymbol{c}, \boldsymbol{n} \rangle \right) \psi \,\mathrm{dl} - \int_C \langle \boldsymbol{c}, \boldsymbol{\nu} \rangle \frac{\partial \psi}{\partial \nu} \,\mathrm{dl} - \left(\langle \boldsymbol{c}, \boldsymbol{t} \rangle \psi \right) |_{\partial C - \partial S}. \tag{12}$$

(c) If $\boldsymbol{F} \in \mathcal{F}(S, \mathbb{R}^3)$, such that $\boldsymbol{F}(\boldsymbol{\phi}) = \int_C \langle \boldsymbol{f}, \partial \boldsymbol{\phi} / \partial \nu \rangle \,\mathrm{dl}$, for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathbb{R}^3)$, where $\boldsymbol{f}$ is a smooth and uniformly bounded vector field on C, then

$$\begin{aligned}
\mathrm{Div}_S\, \boldsymbol{F}(\psi) = \int_C \Bigg(\Bigg(\left\langle \left(\boldsymbol{f} \frac{dk_g}{ds} + k_g \frac{d\boldsymbol{f}}{ds} + (k - k_n) \tau_g \boldsymbol{f} \right), \boldsymbol{t} \right\rangle + \left\langle \left(\boldsymbol{f} {k_g}^2 - \boldsymbol{f} {\tau_g}^2 + k(k - k_n) \boldsymbol{f} \right), \boldsymbol{\nu} \right\rangle \Bigg) \psi + \\
\left\langle \left(\tau_g \frac{d\boldsymbol{f}}{ds} + (k - k_n) k_g \boldsymbol{f} + \boldsymbol{f} \langle \nabla_S \nabla_S \boldsymbol{n}, \boldsymbol{\nu} \otimes \boldsymbol{\nu} \otimes \boldsymbol{\nu} \rangle \right), \boldsymbol{n} \right\rangle \psi + \left(\left\langle \frac{d\boldsymbol{f}}{ds}, \boldsymbol{t} \right\rangle + \langle k_g \boldsymbol{f}, \boldsymbol{\nu} \rangle - \langle 2(k - k_n) \boldsymbol{f}, \boldsymbol{n} \rangle \right) \frac{\partial \psi}{\partial \nu} \Bigg) \mathrm{dl} \\
- \int_C \langle \boldsymbol{f}, \boldsymbol{\nu} \rangle \langle \nabla_S \nabla_S \psi, \boldsymbol{\nu} \otimes \boldsymbol{\nu} \rangle \,\mathrm{dl} - \left(k_g \langle \boldsymbol{f}, \boldsymbol{t} \rangle \psi + \tau_g \langle \boldsymbol{f}, \boldsymbol{n} \rangle \psi + \langle \boldsymbol{f}, \boldsymbol{t} \rangle \frac{\partial \psi}{\partial \nu} \right) \Big|_{\partial C - \partial S}.
\end{aligned} \tag{13}$$

Identities 2.2 can be proved following the proofs for Identities 2.1.

2.5 Existence of potentials

We consider a smooth, oriented, and path-connected surface S whose boundary ∂S is composed of k mutually disjoint connected loops ∂S_i. The set of all mutually disjoint boundary loops is represented as $\mathcal{S} = \{\partial S_i, i = 1 \text{ to } k\}$. The set $\mathcal{S}$ is empty for closed surfaces (such as sphere and torus). A loop on S is *irreducible* if it cannot be continuously deformed into a point without escaping from S. We can associate with surface S a set of n, mutually independent, smooth irreducible loops $\mathcal{L}_S = \{L_i, i = 1 \text{ to } n\}$ such that any irreducible loop on S can be continuously deformed to a linear combination of elements in $\mathcal{L}_S$. By mutual independence we imply that these n loops cannot be continuous deformed into each other (without escaping S). Out of these n loops, there are m loops which can be identified with a subset of boundary loops $\mathcal{S}$; we denote the set of these m irreducible loops as $\mathcal{L}_{\partial S}$. We implicitly assume that the set of mutually independent irreducible boundary loops is necessarily included in $\mathcal{L}_S$. Consider, for instance, a cylindrical surface as shown in Figure 1. There are two mutually disjoint boundary loops ∂S_1 and ∂S_2. There is only one family of irreducible loops on the cylindrical surface; we can identify it with the loop ∂S_1. Therefore $\mathcal{L}_S = \mathcal{L}_{\partial S} = \{\partial S_1\}$ ($n = m = 1$, $k = 2$). For a toroidal surface, as shown in Figure 1, there are no boundary loops ($\mathcal{S} = \emptyset$). There are two independent families of irreducible curves. We denote them by L_1 and L_2. Here $n = 2$ and $m = k = 0$. Finally, for simply connected surfaces, such

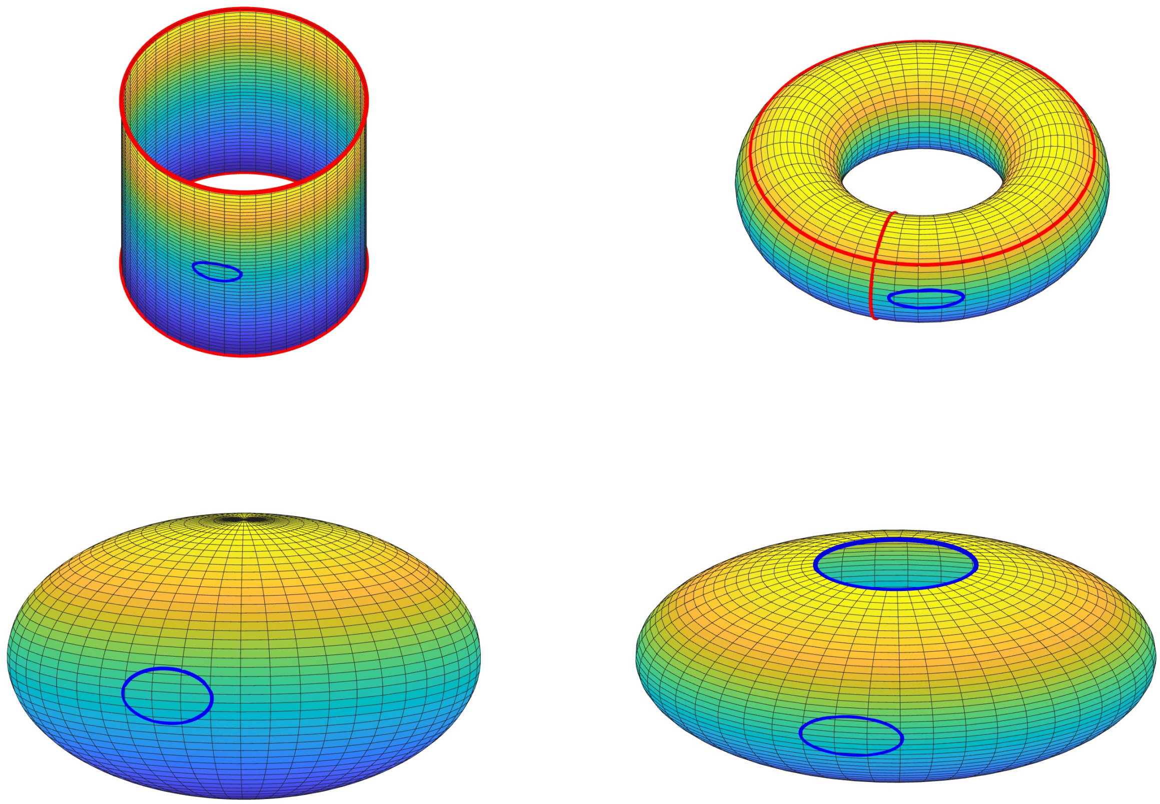

Figure 1: Reducible loops (blue) and irreducible loops (red) on a cylinder, a torus, a sphere, and a sphere with a hole. Cylinder has two disjoint irreducible boundary components which can be continuously deformed to one another. Torus has two independent internal irreducible loops. Sphere has no boundary component and no irreducible loop. Sphere with a hole has a boundary component which forms a reducible loop.

as a sphere or a sphere with a hole (see Figure 1), there are no irreducible loops (although there may be boundary loops).

To motivate the results of this section we begin by recalling the necessary and sufficient conditions on a smooth vector field on S for there to exist a smooth scalar potential such that the vector field is given in terms of the gradient of the scalar potential. Given a smooth vector valued function $\boldsymbol{v} \in C^\infty(S, \mathbb{R}^3)$, there exists a smooth scalar field $u \in C^\infty(S)$, such that $\nabla_S u \times \boldsymbol{n} = \boldsymbol{v}$, if and only if $\int_L \langle \boldsymbol{v}, \boldsymbol{\nu} \rangle \, \mathrm{dl} = 0$ for all loops L on S and $\langle \boldsymbol{v}, \boldsymbol{n} \rangle = 0$ on S. Indeed, given that a smooth u exists, the vector field $\boldsymbol{v} = \nabla_S u \times \boldsymbol{n}$ satisfies both $\int_L \langle \boldsymbol{v}, \boldsymbol{\nu} \rangle \, \mathrm{dl} = 0$ and $\langle \boldsymbol{v}, \boldsymbol{n} \rangle = 0$ identically. On the other hand, given a smooth vector field $\boldsymbol{v}$, which satisfies both the conditions, a potential u can be constructed as $u(\boldsymbol{x}) = \int_{C(\boldsymbol{x})} \langle \boldsymbol{v}, \boldsymbol{\nu} \rangle \, \mathrm{dl}$ where $C(\boldsymbol{x})$ is a smooth curve on S beginning at a fixed point $\boldsymbol{x}_0 \in S$ and ending at $\boldsymbol{x} \in S$. We make three remarks: (i) For a simply connected surface, the necessary and sufficient conditions are equivalent to $\mathrm{div}_S \, \boldsymbol{v} = 0$ and $\langle \boldsymbol{v}, \boldsymbol{n} \rangle = 0$ on S; (ii) If a smooth vector valued function $\boldsymbol{v}$ satisfies $\mathrm{div}_S \, \boldsymbol{v} = 0$, $\langle \boldsymbol{v}, \boldsymbol{n} \rangle = 0$, both on S, and $\int_L \langle \boldsymbol{v}, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 0$ for all $L \in \mathcal{L}_S$, then $\int_L \langle \boldsymbol{v}, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 0$ for any arbitrary loop $L \subset S$. Therefore given a smooth vector valued function $\boldsymbol{v}$, the existence of a smooth scalar function u, satisfying $\nabla_S u = \boldsymbol{v} \times \boldsymbol{n}$, is equivalent to $\operatorname{div}_S \boldsymbol{v} = 0$ and $\langle \boldsymbol{v}, \boldsymbol{n} \rangle = 0$ on S and $\int_L \langle \boldsymbol{v}, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 0$ for all $L \in \mathcal{L}_S$; (iii) An alternate way of stating the result is as follows. Given a smooth vector valued function $\boldsymbol{w} \in C^\infty(S, \mathbb{R}^3)$, there exists a smooth scalar field $u \in C^\infty(S)$, such that $\nabla_S u = \boldsymbol{w}$, if and only if $\int_L \langle \boldsymbol{w}, \boldsymbol{t} \rangle \,\mathrm{dl} = 0$ for all loops L on S and $\langle \boldsymbol{w}, \boldsymbol{n} \rangle = 0$ on S. The latter conditions are equivalent to $\operatorname{div}_S(\boldsymbol{w} \times \boldsymbol{n}) = 0$ and $\langle \boldsymbol{w}, \boldsymbol{n} \rangle = 0$ on S and $\int_L \langle \boldsymbol{w}, \boldsymbol{t} \rangle \,\mathrm{dl} = 0$ for all $L \in \mathcal{L}_S$. The claimed equivalence can be established using $\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{n}$.

The purpose of this section is to generalise the above result for vector fields which are not only piecewise smooth on S but also concentrate over curves $C \subset S$. We will do so by first stating a more general result in terms of distributions.

Lemma 2.3. *Given a distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$, there exists $U \in \mathcal{D}'(S)$ such that $\nabla_S U = \boldsymbol{T}$ if and only if $\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0$ and $\boldsymbol{T}(\boldsymbol{\psi}) = 0$ for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$ such that $\langle \boldsymbol{\psi}, \boldsymbol{n} \rangle = 0$ and $\operatorname{div}_S \boldsymbol{\psi} = 0$.*

The above statement follows from de Rham's theorem A.1; see Appendix A for more details. Before applying the preceding lemma to the problem of our interest, we obtain representations for a compactly supported smooth function $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$ such that $\langle \boldsymbol{\psi}, \boldsymbol{n} \rangle = 0$ and $\operatorname{div}_S \boldsymbol{\psi} = 0$. These representations are useful while translating the distributional result in terms of singular fields for specific cases at hand. While stating the lemma, and in proving it, we use the result that for any $L_i \in \mathcal{L}_S / \mathcal{L}_{\partial S}$ there exists a smooth tangential vector function $\boldsymbol{v}_i : S \to \mathbb{R}^3$ satisfying $\operatorname{div}_S \boldsymbol{v}_i = 0$ and $\int_{L_j} \langle \boldsymbol{v}_i, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 1$ when $i = j$ and 0 when $i \neq j$, keeping in mind that here all $L_j \in \mathcal{L}_S / \mathcal{L}_{\partial S}$, where $\boldsymbol{\nu}$ is the tangential normal to L_j. This is a consequence of the isomorphism between the homology and cohomology groups of S [12].

Lemma 2.4. *Consider a smooth orientable surface $S \subset \mathbb{R}^3$ with $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$ such that $\operatorname{div}_S \boldsymbol{\psi} = 0$ and $\langle \boldsymbol{\psi}, \boldsymbol{n} \rangle = 0$. Then*

1. *If the surface is closed ($\partial S = \emptyset$) there exists $u \in C^\infty(S)$ and n constants $a_i \in \mathbb{R}$ such that $\boldsymbol{\psi} = \nabla_S u \times \boldsymbol{n} + \sum_{i=1}^n a_i \boldsymbol{v}_i$, where n is the number of mutually independent irreducible loops on S.*
2. *If the boundary ∂S is composed of k mutually disjoint connected components ∂S_i and the set $\mathcal{L}_S / \mathcal{L}_{\partial S}$ is empty, i.e., all irreducible loops on S can be expressed as linear combinations of loops in $\mathcal{L}_{\partial S}$, there exists $u \in C^\infty(S)$ such that $\boldsymbol{\psi} = \nabla_S u \times \boldsymbol{n}$ and u is constant at each ∂S_i that is $u = b_i$ at ∂S_i.*

The above representations exclude surfaces where both $\mathcal{L}_S / \mathcal{L}_{\partial S}$ and $\mathcal{L}_{\partial S}$ are non-empty. We do so to avoid a considerably more involved proof and also due to the fact that the surfaces covered by the above representations are sufficient for the discussion at hand.

Proof. 1. Let $a_i = \int_{L_i} \langle \boldsymbol{\psi}, \boldsymbol{\nu} \rangle \,\mathrm{dl}$. The smooth function $\bar{\boldsymbol{\psi}} \in C^\infty(S, \mathbb{R}^2)$, defined as $\bar{\boldsymbol{\psi}} = \boldsymbol{\psi} - \sum_{i=1}^n a_i \boldsymbol{v}_i$, satisfies $\int_L \langle \bar{\boldsymbol{\psi}}, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 0$ for all loops $L \subset S$ implying the existence of $u \in C^\infty(S)$ such that $\bar{\boldsymbol{\psi}} = \nabla_S u \times \boldsymbol{n}$. 2. That $\boldsymbol{\psi}$ satisfies $\int_L \langle \boldsymbol{\psi}, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 0$ for all loops $L \subset S$ implies the existence of $u \in C^\infty(S)$ such that $\bar{\boldsymbol{\psi}} = \nabla_S u \times \boldsymbol{n}$. Due to the compactly supported nature of $\boldsymbol{\psi}$, u is constant on each connected component ∂S_i of the boundary. □

We consider three cases to illustrate applications of the above results. For brevity we restrict our attention to curves C such that $\partial C - \partial S = \emptyset$, i.e., the curves either end on the boundary of the surface or they form a closed loop. They cannot end in the interior of the surface. In a remark, which follows the three cases, we discuss the situation where curves are allowed to end at an interior point in S.

Case (i): (*S is simply connected*) For a simply connected surface S, given any $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$ such that $\langle \boldsymbol{\psi}, \boldsymbol{n} \rangle = 0$ and $\operatorname{div}_S \boldsymbol{\psi} = 0$ on S, there exists $\phi \in \mathcal{D}(S)$ such that $\nabla_S \phi \times \boldsymbol{n} = \boldsymbol{\psi}$. Consequently, given $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$, the necessary and sufficient conditions for the existence of $U \in \mathcal{D}'(\Omega)$ such that $\nabla_S U = \boldsymbol{T}$ are

$$\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0 \text{ and } \operatorname{Div}_S(\boldsymbol{T} \times \boldsymbol{n}) = 0. \tag{14}$$

To elaborate further, consider a distribution of the form

$$\boldsymbol{T}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle \, \mathrm{da} + \int_C \langle \boldsymbol{c}, \boldsymbol{\psi} \rangle \, \mathrm{dl} + \int_C \langle \boldsymbol{f}, \partial \boldsymbol{\psi} / \partial \nu \rangle \, \mathrm{dl}, \tag{15}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$, where $\boldsymbol{b}$ is a piecewise smooth and uniformly bounded vector field on S (possibly discontinuous across C), $\boldsymbol{c}$ is a smooth and uniformly bounded vector field on C, and $\boldsymbol{f}$ is a smooth and uniformly bounded vector field on C. This distribution generalises the idea of a smooth field on S to allow for discontinuities and concentrations. With (15) in mind, we use Identities 2.2 and the notion of multiplying a distribution with a smooth field to localise the distributional equations $\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0$ and $\operatorname{Div}_S(\boldsymbol{T} \times \boldsymbol{n}) = 0$ in terms of the following equivalent point wise conditions:

$$\langle \boldsymbol{b}, \boldsymbol{n} \rangle = 0 \text{ and } \operatorname{div}_S(\boldsymbol{b} \times \boldsymbol{n}) = 0, \text{ on } S/C, \tag{16a}$$

$$\boldsymbol{f} = \langle \boldsymbol{f}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}, \tag{16b}$$

$$\langle \boldsymbol{c}, \boldsymbol{n} \rangle - (k - k_n) \langle \boldsymbol{f}, \boldsymbol{\nu} \rangle = 0, \tag{16c}$$

$$\langle \boldsymbol{c}, \boldsymbol{t} \rangle + \left\langle \frac{d\boldsymbol{f}}{ds}, \boldsymbol{\nu} \right\rangle = 0, \text{ and} \tag{16d}$$

$$\left\langle \frac{d\boldsymbol{c}}{ds}, \boldsymbol{\nu} \right\rangle + (k - k_n) \left\langle \frac{d\boldsymbol{f}}{ds}, \boldsymbol{n} \right\rangle + \frac{dk_g}{ds} \langle \boldsymbol{f}, \boldsymbol{\nu} \rangle + 2k_g \left\langle \frac{d\boldsymbol{f}}{ds}, \boldsymbol{\nu} \right\rangle + \langle [\![\boldsymbol{b}]\!], \boldsymbol{t} \rangle = 0 \text{ on } C. \tag{16e}$$

The classical conditions are immediate when $[\![\boldsymbol{b}]\!] = \boldsymbol{0}$, $\boldsymbol{c} = \boldsymbol{0}$, and $\boldsymbol{f} = \boldsymbol{0}$. On the other hand, if only the higher order concentration field $\boldsymbol{f}$ is absent, then it follows that the concentration $\boldsymbol{c}$ is such that $\boldsymbol{c} = \langle \boldsymbol{c}, \boldsymbol{\nu} \rangle \boldsymbol{\nu}$ and that $\left\langle \frac{d\boldsymbol{c}}{ds}, \boldsymbol{\nu} \right\rangle + \langle [\![\boldsymbol{b}]\!], \boldsymbol{t} \rangle = 0$, both on C. We recall that Equations (16) are both necessary and sufficient for the distribution in (15) to be written as a surface gradient of a scalar potential on a simply connected surface S. The regularity of the potential will be discussed in the next section.

Case (ii): (*S is topologically equivalent to a cylinder*) A surface S, topologically equivalent to a cylinder, has two mutually disjoint boundary curves (∂S_1 and ∂S_2); the set $\mathcal{L}_S = \mathcal{L}_{\partial S}$ consists of one irreducible loop which can be identified with any one of the two boundary loops (say ∂S_1). We can use the second part of Lemma 2.4 for the case at hand. Consequently, for $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$, such that $\langle \boldsymbol{\psi}, \boldsymbol{n} \rangle = 0$ and $\operatorname{div}_S \boldsymbol{\psi} = 0$, there exists $\phi \in C^\infty(S)$ such that $\nabla_S \phi \times \boldsymbol{n} = \boldsymbol{\psi}$. Note that, unlike Case (i) above, ϕ is not necessarily compactly supported. Accordingly, conditions (14) are necessary but no longer sufficient for the existence of a potential $U \in \mathcal{D}'(S)$ such that $\nabla_S U = \boldsymbol{T}$. For a distribution $\boldsymbol{T}$ of the form (15) we use Lemma 2.3

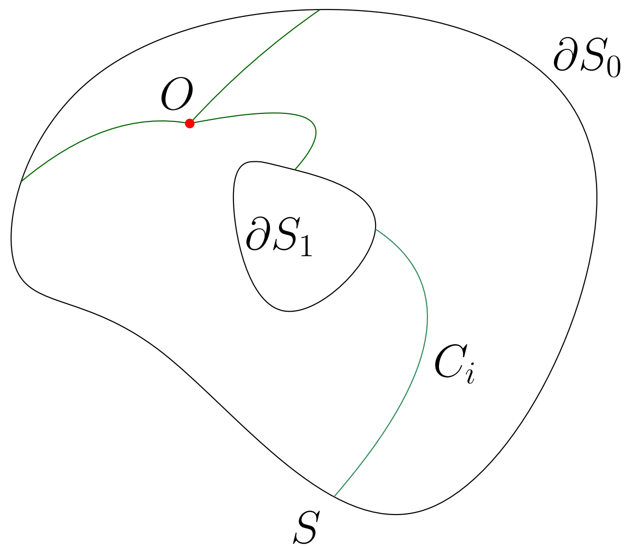

Figure 2: A curved surface S with two mutually disjoint boundary curves ∂S_1 and ∂S_2, several singular curves C_i and a junction point O.

to derive the necessary and sufficient conditions. The local conditions (16) are obtained as conjugate to the local value of ϕ in the interior of the surface whereas an additional global condition appears as the conjugate to the constant valued ϕ at one of the boundary components. The global condition is of the form

$$\int_{\partial S_1} \langle \boldsymbol{b}, \boldsymbol{t} \rangle \,\mathrm{dl} + \langle \boldsymbol{c}, \boldsymbol{\nu} \rangle|_{C \cap \partial S_1} + k_g \langle \boldsymbol{f}, \boldsymbol{\nu} \rangle|_{C \cap \partial S_1} = 0, \tag{17}$$

where $\boldsymbol{t}$ is the unit tangent to ∂S_1 and $\boldsymbol{\nu}$ is the in-plane normal to C at $C \cap \partial S_1$. Altogether (16) and (17) are the required necessary and sufficient conditions.

Case (iii): (S *is topologically equivalent to a torus*) A toroidal surface S is closed ($\partial S = \emptyset$) and has two mutually independent irreducible loops, say L_1 and L_2. We can use the first part of Lemma 2.4 for the present case. The two smooth tangential vector fields on S, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, which appear in the Lemma, are such that $\int_{L_1} \langle \boldsymbol{v}_1, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 1$, $\int_{L_2} \langle \boldsymbol{v}_1, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 0$, $\int_{L_2} \langle \boldsymbol{v}_2, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 1$, $\int_{L_1} \langle \boldsymbol{v}_2, \boldsymbol{\nu} \rangle \,\mathrm{dl} = 0$, $\mathrm{div}_S\, \boldsymbol{v}_1 = 0$, and $\mathrm{div}_S\, \boldsymbol{v}_2 = 0$. According to the Lemma, for any $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$ satisfying $\langle \boldsymbol{\psi}, \boldsymbol{n} \rangle = 0$ and $\mathrm{div}_S\, \boldsymbol{\psi} = 0$ on S, there exists $\phi \in C^\infty(S)$ and $a_1, a_2 \in \mathbb{R}$ such that $\nabla_S \phi \times \boldsymbol{n} + a_1 \boldsymbol{v}_1 + a_2 \boldsymbol{v}_2 = \boldsymbol{\psi}$. With this representation for $\boldsymbol{\psi}$, and using Lemma 2.3, we can derive the necessary and sufficient conditions, for a distribution $\boldsymbol{T}$ (on the torus S) of the form (15) to be given in terms of a scalar potential $U \in \mathcal{D}'(S)$ such that $\nabla_S U = \boldsymbol{T}$, as given by (14) (or equivalently as (16)) in addition to the following global conditions:

$$\int_{L_i} \langle \boldsymbol{b}, \boldsymbol{t} \rangle \,\mathrm{dl} + \langle \boldsymbol{c}, \boldsymbol{\nu} \rangle|_{C \cap L_i} + k_g \langle \boldsymbol{f}, \boldsymbol{\nu} \rangle|_{C \cap L_i} = 0 \text{ for } i = 1, 2, \tag{18}$$

where $\boldsymbol{t}$ is the unit tangent to L_i and $\boldsymbol{\nu}$ is the in-plane normal to C at $C \cap \partial L_i$. The global conditions appear as conjugate to the coefficients a_i.

Remark 2.1. (Multiple interfaces and a junction point) Let S be a surface topologically equivalent to a cylinder with two mutually disjoint boundary curves ∂S_1 and ∂S_2, see Figure 2. Consider ∂S_1 to be the only element of the set $\mathcal{L}_S = \mathcal{L}_{\partial S}$. Let $C_i \subset S$ denote m curves on S such that their end points are either on the boundary ∂S or at an interior point $O \in S$, i.e., $\partial C_i \subset \{O\} \cup \partial S$. The point O is the junction of interfaces in the interior of the domain. Let $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ be a distribution on S such that

$$\boldsymbol{T}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle \,\mathrm{da} + \sum_{i=1}^{m} \int_{C_i} \langle \boldsymbol{c}_i, \boldsymbol{\psi} \rangle \,\mathrm{dl} + \sum_{i=1}^{m} \int_{C_i} \langle \boldsymbol{f}_i, \partial \boldsymbol{\psi}/\partial \nu \rangle \,\mathrm{dl}, \tag{19}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$, where $\boldsymbol{b}$ is a piecewise smooth and uniformly bounded vector field on S (possibly discontinuous across C_i), $\boldsymbol{c}_i$ are smooth and uniformly bounded vector fields on C_i, respectively, and $\boldsymbol{f}_i$ are smooth and uniformly bounded vector fields on C_i, respectively. The necessary and sufficient conditions for the existence of $U \in \mathcal{D}'(\Omega)$ such that $\nabla_S U = \boldsymbol{T}$ are

$$\langle \boldsymbol{b}, \boldsymbol{n} \rangle = 0 \text{ and } \mathrm{div}_S(\boldsymbol{b} \times \boldsymbol{n}) = 0, \text{ on } S/\{C_i \cup O\}, \tag{20a}$$

$$\boldsymbol{f}_i = \langle \boldsymbol{f}_i, \boldsymbol{\nu} \rangle \boldsymbol{\nu},\ \langle \boldsymbol{c}_i, \boldsymbol{n} \rangle - (k - k_n)\langle \boldsymbol{f}_i, \boldsymbol{\nu} \rangle = 0,\ \langle \boldsymbol{c}_i, \boldsymbol{t} \rangle + \left\langle \frac{d\boldsymbol{f}_i}{ds}, \boldsymbol{\nu} \right\rangle = 0, \text{ and} \tag{20b}$$

$$\left\langle \frac{d\boldsymbol{c}_i}{ds}, \boldsymbol{\nu} \right\rangle + (k - k_n) \left\langle \frac{d\boldsymbol{f}_i}{ds}, \boldsymbol{n} \right\rangle + \frac{dk_g}{ds} \langle \boldsymbol{f}_i, \boldsymbol{\nu} \rangle + 2k_g \left\langle \frac{d\boldsymbol{f}_i}{ds}, \boldsymbol{\nu} \right\rangle + \langle [\![\boldsymbol{b}]\!], \boldsymbol{t} \rangle = 0 \text{ on } C_i/O, \tag{20c}$$

$$\sum_{i=1}^{m} \big(\langle \boldsymbol{c}_i, \boldsymbol{\nu}_i \rangle + k_{g_i} \langle \boldsymbol{f}_i, \boldsymbol{\nu}_i \rangle \big) = 0 \text{ and } \sum_{i=1}^{m} \langle \boldsymbol{f}_i, \boldsymbol{\nu}_i \rangle = 0 \text{ at } O, \text{ and} \tag{20d}$$

$$\int_{\partial S_1} \langle \boldsymbol{b}, \boldsymbol{t} \rangle \,\mathrm{dl} + \sum_{i=1}^{m} \langle \boldsymbol{c}_i, \boldsymbol{\nu}_i \rangle|_{C_i \cap \partial S_1} + \sum_{i=1}^{m} k_{g_i} \langle \boldsymbol{f}_i, \boldsymbol{\nu}_i \rangle|_{C_i \cap \partial S_1} = 0. \tag{20e}$$

In Equations (20d), k_{g_i} and $\boldsymbol{\nu}_i$ denote the geodesic curvature of C_i and the in-plane normal to C_i, respectively, evaluated at O, whereas in Equation (20e) they are evaluated at $C_i \cap \partial S_1$.

2.6 Regularity of potentials

In the preceding section we derived the necessary and sufficient conditions on a distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ for there to exist a potential $U \in \mathcal{D}'(\Omega)$ such that $\nabla_S U = \boldsymbol{T}$. In particular, we emphasised the role of the regularity restrictions on $\boldsymbol{T}$ and the topological assumptions on S on the derivation of the point wise equivalent form of these conditions. In this section we discuss the regularity of the potential U given the regularity of $\boldsymbol{T}$. Our first result pertains to simply connected surfaces S which are topologically equivalent to a disc.

Lemma 2.5. *Consider a regular, oriented surface S topologically equivalent to a disc and a regular curve C such that $\partial C - \partial S = \emptyset$.*

1. *Given a distribution $\boldsymbol{I} \in \mathcal{I}(S, \mathbb{R}^3)$ such that $\langle \boldsymbol{I}, \boldsymbol{n} \rangle = 0$ and $\mathrm{Div}_S(\boldsymbol{I} \times \boldsymbol{n}) = 0$, there exists $U \in \mathcal{B}(S)$ such that $\nabla_S U = \boldsymbol{I}$.*

2. *Given a distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ such that $\boldsymbol{T} = \boldsymbol{B} + \boldsymbol{I}$ where $\boldsymbol{B} \in \mathcal{B}(S, \mathbb{R}^3)$ and $\boldsymbol{I} \in \mathcal{I}(S, \mathbb{R}^3)$ satisfying $\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0$ and $\mathrm{Div}_S(\boldsymbol{T} \times \boldsymbol{n}) = 0$, there exists $U \in \mathcal{B}(S)$ such that $\nabla_S U = \boldsymbol{T}$.*

3. *Given a distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ such that $\boldsymbol{T} = \boldsymbol{B} + \boldsymbol{I} + \boldsymbol{F}$ where $\boldsymbol{B} \in \mathcal{B}(S, \mathbb{R}^3)$, $\boldsymbol{I} \in \mathcal{I}(S, \mathbb{R}^3)$ and $\boldsymbol{F} \in \mathcal{F}(S, \mathbb{R}^3)$ satisfying $\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0$ and $\mathrm{Div}_S(\boldsymbol{T} \times \boldsymbol{n}) = 0$, there exists $U \in \mathcal{D}'(S)$ such that $U = U_1 + U_2$ where $U_1 \in \mathcal{B}(S)$ and $U_2 \in \mathcal{I}(S)$ such that $\nabla_S U = \boldsymbol{T}$.*

Proof. Given S is topologically equivalent to a disc, and that C satisfies $\partial C - \partial S = \emptyset$, the curve C divides S into two regular oriented surfaces S^+ and S^- such that $S = S^+ \cup S^-$ and $\partial S^+ \cap \partial S^- = C$.

1. Consider a smooth vector field $\boldsymbol{c} : C \to \mathbb{R}^3$ such that $\boldsymbol{I}(\boldsymbol{\psi}) = \int_C \langle \boldsymbol{c}, \boldsymbol{\psi} \rangle \,\mathrm{dl}$, for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$. Use Identity (12) with $\langle \boldsymbol{I}, \boldsymbol{n} \rangle = 0$ and $\mathrm{Div}_S(\boldsymbol{I} \times \boldsymbol{n}) = 0$ to infer that $\boldsymbol{c} = \alpha \boldsymbol{\nu}$ where α is constant on C. A distribution $U \in \mathcal{B}(S)$ defined by $U(\psi) = \int_{S^+} \alpha \psi \,\mathrm{da}$, for all $\psi \in \mathcal{D}(S)$, satisfies $\nabla_S U = \boldsymbol{I}$.

2. For all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$, let $\boldsymbol{B}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle \,\mathrm{da}$ and $\boldsymbol{I}(\boldsymbol{\psi}) = \int_C \langle \boldsymbol{c}, \boldsymbol{\psi} \rangle \,\mathrm{dl}$. Then $\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0$ and $\mathrm{Div}_S(\boldsymbol{T} \times \boldsymbol{n}) = 0$ imply that $\langle \boldsymbol{b}, \boldsymbol{n} \rangle = 0$ and $\mathrm{div}_S(\boldsymbol{b} \times \boldsymbol{n}) = 0$ in S/C and $\boldsymbol{c} = \alpha \boldsymbol{\nu}$, $\langle [\![\boldsymbol{b}]\!], \boldsymbol{t} \rangle = -d\alpha/ds$ on C, where α is a scalar field on C. According to the first two implications, there exists smooth fields $u_1 : S^+ \to \mathbb{R}$ and $u_2 : S^- \to \mathbb{R}$ such that $\nabla_S u_1 = \boldsymbol{b}$ in S^+ and $\nabla_S u_2 = \boldsymbol{b}$ in S^-. We define $U_1 \in \mathcal{B}(S)$ such that $U_1(\psi) = \int_{S^+} u_1 \psi \,\mathrm{da} + \int_{S^-} u_2 \psi \,\mathrm{da}$, for all $\psi \in \mathcal{D}(S)$. We calculate $\nabla_S U_1(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle \,\mathrm{da} - \int_C (u_1 - u_2) \langle \boldsymbol{\psi}, \boldsymbol{\nu} \rangle \,\mathrm{dl}$. Therefore, $(\boldsymbol{T} - \nabla_S U_1) \in \mathcal{I}(S, \mathbb{R}^3)$. The first part of the present lemma can then be used to establish the existence of $U \in \mathcal{B}(S)$ such that $\nabla_S U = \boldsymbol{T}$.

3. For all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$, let $\boldsymbol{B}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle \,\mathrm{da}$, $\boldsymbol{I}(\boldsymbol{\psi}) = \int_C \langle \boldsymbol{c}, \boldsymbol{\psi} \rangle \,\mathrm{dl}$, and $\boldsymbol{F}(\boldsymbol{\psi}) = \int_C \langle \boldsymbol{f}, \partial \boldsymbol{\psi}/\partial \nu \rangle \,\mathrm{dl}$. Then using conditions $\langle \boldsymbol{T}, \boldsymbol{n} \rangle$ and $\mathrm{Div}_S(\boldsymbol{T} \times \boldsymbol{n}) = 0$ we can show that $\boldsymbol{f} = f \boldsymbol{\nu}$, where f is a smooth scalar field on C. Define $U_1 \in \mathcal{I}(S)$ such that $U_1(\psi) = -\int_C f \psi \,\mathrm{dl}$, for all $\psi \in \mathcal{D}(S)$. We calculate, using Identity (9), $\nabla_S U_1(\boldsymbol{\psi}) = -\int_C \left\langle \left(\frac{df}{ds} \boldsymbol{t} + \kappa_g f \boldsymbol{\nu} - f(k - \kappa_n) \boldsymbol{n} \right), \boldsymbol{\psi} \right\rangle \mathrm{dl} + \int_C \langle \boldsymbol{f}, \partial \boldsymbol{\psi}/\partial \nu \rangle \,\mathrm{dl}$. Therefore, $(\boldsymbol{T} - \nabla_S U_1) = \boldsymbol{B} + \boldsymbol{I} + \boldsymbol{I}_1$, where $\boldsymbol{I}_1 \in \mathcal{I}(S)$. Using the second part of the present lemma, we can infer existence of a $U_2 \in \mathcal{B}(S)$ such that $(\boldsymbol{T} - \nabla_S U_1) = \nabla_S U_2$. The distribution $U = (U_1 + U_2)$ satisfies the required conditions.

□

With Lemma 2.5 in place, we can now proceed to establish analogous regularity results for arbitrary topological surfaces.

Lemma 2.6. *Consider a regular, oriented surface $S \subset \mathbb{R}^3$ and a regular curve $C \subset S$ such that $\partial C - \partial S = \emptyset$.*

1. *Given a distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ such that $\boldsymbol{T} = \boldsymbol{B} + \boldsymbol{I}$, where $\boldsymbol{B} \in \mathcal{B}(S, \mathbb{R}^3)$ and $\boldsymbol{I} \in \mathcal{I}(S, \mathbb{R}^3)$, satisfying $\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0$ and $\boldsymbol{T}(\boldsymbol{\psi}) = 0$ for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$ such that $\langle \boldsymbol{\psi}, \boldsymbol{n} \rangle = 0$ and $\mathrm{div}_S(\boldsymbol{\psi}) = 0$, there exists $U \in \mathcal{B}(S)$ such that $\nabla_S U = \boldsymbol{T}$.*

2. *Given a distribution $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ such that $\boldsymbol{T} = \boldsymbol{B} + \boldsymbol{I} + \boldsymbol{F}$ where $\boldsymbol{B} \in \mathcal{B}(S, \mathbb{R}^3)$, $\boldsymbol{I} \in \mathcal{I}(S, \mathbb{R}^3)$ and $\boldsymbol{F} \in \mathcal{F}(S, \mathbb{R}^3)$ satisfying $\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0$ and $\mathrm{Div}_S(\boldsymbol{T} \times \boldsymbol{n}) = 0$, there exists $U \in \mathcal{D}'(S)$ such that $U = U_1 + U_2$ where $U_1 \in \mathcal{B}(S)$ and $U_2 \in \mathcal{I}(S)$ such that $\nabla_S U = \boldsymbol{T}$.*

Proof. Let $\{S_\alpha\}$ denote an open cover of S such that each S_α is topologically equivalent to a disc. Let g_α be a partition of unity subordinate to this open cover.

1. For $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ satisfying the given conditions, there exists a $V \in \mathcal{D}'(S)$ such that $\nabla_S V = \boldsymbol{T}$. For any S_α, the regularity result in the second part of Lemma 2.5 implies the existence of a piecewise

smooth map $u_\alpha : S_\alpha \to \mathbb{R}$ such that $V|_{S_\alpha}(\psi) = \int_{S_\alpha} u_\alpha \psi \, \mathrm{da}$, for all $\psi \in \mathcal{D}(S)$. Let $u = u_\alpha g_\alpha$. Then $U \in \mathcal{B}(S)$, with $U(\psi) = \int_S u\psi \, \mathrm{da}$, satisfies $\nabla_S U = \boldsymbol{T}$.

2. For $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ satisfying the given conditions, there exists a $V \in \mathcal{D}'(S)$ such that $\nabla_S V = \boldsymbol{T}$. For any S_α, the regularity result in the third part of Lemma 2.5 implies the existence of a piecewise smooth map $u_\alpha : S_\alpha \to \mathbb{R}$ and $\bar{u}_\alpha : S_\alpha \cap C \to \mathbb{R}$ such that $V|_{S_\alpha}(\psi) = \int_{S_\alpha} u_\alpha \psi \, \mathrm{da} + \int_{S_\alpha \cap C} \bar{u}_\alpha \psi \, \mathrm{dl}$, for all $\psi \in \mathcal{D}(S)$. Let $u : S \to \mathbb{R}$ and $\bar{u} : C \to \mathbb{R}$ be such that $u = u_\alpha g_\alpha$ and $\bar{u} = \bar{u}_\alpha g_\alpha$, respectively. Then $U \in \mathcal{D}'(S)$, with $U = U_1 + U_2$, $U_1 \in \mathcal{B}(S)$, $U_2 \in \mathcal{I}(S)$, such that $U_1(\psi) = \int_S u\psi \, \mathrm{da}$ and $U_1(\psi) = \int_C \bar{u}\psi \, \mathrm{dl}$, satisfies $\nabla_S U = \boldsymbol{T}$.

□

The potentials discussed above are unique upto a constant. Indeed, if for a given $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$ there exists two distributions $U \in \mathcal{D}'(S)$ and $\tilde{U} \in \mathcal{D}'(S)$ such that $\nabla_S U = \boldsymbol{T}$ and $\nabla_S \tilde{U} = \boldsymbol{T}$ then the distribution $\tilde{W} = U - \tilde{U}$ satisfies $\nabla_S W = \boldsymbol{0}$. Therefore W is necessarily constant (see Appendix B for a proof).

3 Strain compatibility for linear shells

In this section we use the mathematical infrastructure developed in the preceding section to derive the strain compatibility conditions for singular strain fields on a shell with linearised kinematics. A smooth, oriented, path-connected surface $S \subset \mathbb{R}^3$, as considered in the mathematical preliminaries, is taken as the reference configuration for the shell surface. In the first subsection below, we collect the pointwise strain-displacement relationships both in the bulk of the shell surface and on the singular curves (and their junctions) in the interior of the surface. We allow for the displacement field to be discontinuous across the curve (but smooth otherwise), thus engendering concentrations in stretching and bending strain fields. In the second subsection, we derive the necessary and sufficient (compatibility) conditions for such singular strain fields, i.e., with discontinuities and concentrations, to be expressible in terms of displacements as posited in the first subsection. These conditions, which are the main result of this paper, demonstrate the interplay between the shell geometry, its topology, and the non-smooth deformation kinematics.

3.1 Strain-displacement relations

Given a reference surface $S \subset \mathbb{R}^3$ that undergoes a *smooth* and uniformly bounded displacement $\boldsymbol{u} : S \to \mathbb{R}^3$, the linearised smooth stretching strain field $\boldsymbol{e} : S \to \mathrm{Sym}$ and smooth bending strain field $\boldsymbol{\lambda} : S \to \mathrm{Sym}$ are

$$\boldsymbol{e} = \mathbb{P}\,\mathrm{sym}(\nabla_S \boldsymbol{u})\mathbb{P} \text{ and} \tag{21a}$$

$$\boldsymbol{\lambda} = \mathbb{P}(\boldsymbol{n}\nabla_S\nabla_S \boldsymbol{u})\mathbb{P}, \tag{21b}$$

where $\boldsymbol{n}\nabla_S\nabla_S\boldsymbol{u} \in \mathrm{Lin}$ is defined such that, for any $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^3$, $\langle \boldsymbol{n}\nabla_S\nabla_S\boldsymbol{u}, \boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \rangle = \langle \nabla_S\nabla_S\boldsymbol{u}, \boldsymbol{n} \otimes \boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \rangle$. While the stretching strain field $\boldsymbol{e}$ measures the change in the length of tangent vectors on the shell surface, the bending strain field $\boldsymbol{\lambda}$ measures the change in the curvature of the shell surface, both with respect to the reference surface S.

A *singular* displacement field is represented in terms of a distribution $\boldsymbol{U} \in \mathcal{D}'(S, \mathbb{R}^3)$. Introducing distributional representations of stretching and bending strains as $\boldsymbol{E} \in \mathcal{D}'(S, \text{Sym})$ and $\boldsymbol{\Lambda} \in \mathcal{D}'(S, \text{Sym})$, respectively, we posit the distributional strain-displacement relations as

$$\boldsymbol{E} = \mathbb{P}\,\text{sym}(\nabla_S \boldsymbol{U})\mathbb{P} \text{ and} \tag{22a}$$

$$\boldsymbol{\Lambda} = \mathbb{P}(\boldsymbol{n}\nabla_S\nabla_S \boldsymbol{U})\mathbb{P}. \tag{22b}$$

For a sequence of smooth displacement fields that converge to a singular displacement field (in a weak sense), the smooth strain fields would converge to distributional strains consistent with our definition of the singular strain fields in (22). In the following, we will restrict our attention to two cases. First, we consider displacement fields which are piecewise smooth and continuous in S. The gradients in the displacement field are allowed to be discontinuous across m interfacial curves $C_i \in S$ and their junction point O such that $\partial C_i \subset \{O\} \cup \partial S$. We call such interfaces *perfectly bonded.* Second, we consider displacement fields which are piecewise smooth, but not necessarily continuous, in S. The discontinuities in the displacement, as well as its gradients, are permissible across the m interfacial curves $C_i \in S$. We call such interfaces *imperfectly bonded.*

(i) Perfectly bonded interfaces. Let $\boldsymbol{u}$ be a displacement vector field uniformly bounded on S, smooth over $S/\cup_{i=1}^m C_i$ and continuous across $C_i \subset S$. Therefore, at any singular interface C_i, tearing and slipping is prohibited but a fold is allowed; the rest of the surface deforms smoothly. Given piecewise smooth and continuous $\boldsymbol{u}$, we introduce a distributional displacement $\boldsymbol{U} \in \mathcal{B}(S, \mathbb{R}^3)$ such that

$$\boldsymbol{U}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{u}, \boldsymbol{\psi} \rangle \,\text{da}, \tag{23}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(S, \text{Lin})$. The corresponding stretching and bending strain distributions follow from (22) and Identities 2.1. We obtain

$$\boldsymbol{E}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{e}, \boldsymbol{\psi} \rangle \,\text{da} \;\text{ and} \tag{24a}$$

$$\boldsymbol{\Lambda}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{\lambda}, \boldsymbol{\psi} \rangle \,\text{da} + \sum_{i=1}^m \int_{C_i} \langle \boldsymbol{\gamma}, \boldsymbol{\psi} \rangle \,\text{dl}, \tag{24b}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$, where $\boldsymbol{e}$ and $\boldsymbol{\lambda}$ are as given in (21) (but now defined over $S/\cup_{i=1}^m C_i$). The interfacial concentration of bending strain, given by $\boldsymbol{\gamma} : C_i \to \text{Sym}$, is a smooth field on C_i:

$$\boldsymbol{\gamma} = -\left\langle \boldsymbol{n}, \left[\!\left[\frac{\partial \boldsymbol{u}}{\partial \nu} \right]\!\right] \right\rangle \boldsymbol{\nu} \otimes \boldsymbol{\nu}. \tag{25}$$

We take presence of the interfacial concentration of bending strain synonymous with a *fold* at C_i. The interfacial strain $\boldsymbol{\gamma}$ essentially measures the strength of the fold at the interface as a result of the deformation. Altogether, for a shell with perfectly bonded interfaces, the strain-displacement relations are given by (21), defined on $S/\cup_{i=1}^m C_i$, and (25) on $\cup_{i=1}^m C_i$.

(ii) Imperfectly bonded interfaces. The displacement field is now allowed to be discontinuous across $\cup_{i=1}^m C_i$, but smooth everywhere else on S. Accordingly, we permit tearing and slipping across an interface on the shell surface in addition to folding. Given such a displacement field, we can introduce a distributional displacement $\boldsymbol{U} \in \mathcal{B}(\Omega, \mathbb{R}^3)$ of the form (23). The corresponding stretching and bending strain distributions follow from (22) and Identities 2.1 as

$$\boldsymbol{E}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{e}, \boldsymbol{\psi} \rangle \,\mathrm{da} + \sum_{i=1}^m \int_{C_i} \langle \boldsymbol{e}_C, \boldsymbol{\psi} \rangle \,\mathrm{dl} \;\text{ and} \tag{26a}$$

$$\boldsymbol{\Lambda}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{\lambda}, \boldsymbol{\psi} \rangle \,\mathrm{da} + \sum_{i=1}^m \int_{C_i} \langle \boldsymbol{\gamma}, \boldsymbol{\psi} \rangle \,\mathrm{dl} + \sum_{i=1}^m \int_{C_i} \left\langle \boldsymbol{\gamma}_1, \frac{\partial \boldsymbol{\psi}}{\partial \nu} \right\rangle \mathrm{dl} + \langle \boldsymbol{\gamma}_O, \boldsymbol{\psi}(O) \rangle, \tag{26b}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathrm{Lin})$, where the bulk fields $\boldsymbol{e}$ and $\boldsymbol{\lambda}$, piecewise smooth on S, are given by (21) for all positions away from C_i. The interfacial concentration of stretching strain, given by $\boldsymbol{e}_C : \cup_{i=1}^m C_i \to \mathrm{Sym}$, is a smooth field on each interface C_i given in terms of the tangential component of the jump in $\boldsymbol{u}$ across C_i:

$$\boldsymbol{e}_C = -\frac{1}{2} \left(\mathbb{P}[\![\boldsymbol{u}]\!] \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \mathbb{P}[\![\boldsymbol{u}]\!] \right). \tag{27}$$

The interfacial stretching strain $\boldsymbol{e}_C$ measures the in-plane *tearing* at the interface. The interfacial concentrations in bending strain, given by $\boldsymbol{\gamma} : \cup_{i=1}^m C_i \to \mathrm{Sym}$ and $\boldsymbol{\gamma}_1 : \cup_{i=1}^m C_i \to \mathrm{Sym}$, are smooth fields on each C_i:

$$\begin{aligned} \boldsymbol{\gamma} = -\left\langle \boldsymbol{n}, \left[\!\!\left[\frac{\partial \boldsymbol{u}}{\partial s} \right]\!\!\right] \right\rangle (\boldsymbol{t} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{t}) - \left(\left\langle \boldsymbol{n}, \left[\!\!\left[\frac{\partial \boldsymbol{u}}{\partial \nu} \right]\!\!\right] \right\rangle + \tau_g \langle [\![\boldsymbol{u}]\!], \boldsymbol{t} \rangle + (k - k_n) \langle [\![\boldsymbol{u}]\!], \boldsymbol{\nu} \rangle \right) (\boldsymbol{\nu} \otimes \boldsymbol{\nu}) \\ + k_g \langle [\![\boldsymbol{u}]\!], \boldsymbol{n} \rangle (\boldsymbol{t} \otimes \boldsymbol{t} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}) + (k - k_n) \langle [\![\boldsymbol{u}]\!], \boldsymbol{n} \rangle (\boldsymbol{n} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{n}) \;\text{ and} \end{aligned} \tag{28a}$$

$$\boldsymbol{\gamma}_1 = \langle \boldsymbol{n}, [\![\boldsymbol{u}]\!] \rangle \, \boldsymbol{\nu} \otimes \boldsymbol{\nu}. \tag{28b}$$

We can continue to interpret the interfacial bending strain $\boldsymbol{\gamma}$ as the fold strength, a notion which is not immediate for an imperfectly bonded interface. Indeed, there is no fold, as we would like to think intuitively, at the interface which has also experience tear. The interfacial strain $\boldsymbol{\gamma}$ is affected both by the in-plane and the out-of-plane tearing at the interface. The interfacial bending strain $\boldsymbol{\gamma}_1$ measures the out-of-plane tearing at the interface. The discontinuity of the deformation therefore produces a monopole concentration in the stretching strain, a monopole concentration in the bending strain, and a dipole concentration in the bending strain. We can accordingly view $\boldsymbol{\gamma}_1$ as a *fold dipole* at C_i. The distributional bending strain in (26b) also carries a point concentration in the bending strain $\boldsymbol{\gamma}_O \in \mathrm{Sym}$ given by

$$\boldsymbol{\gamma}_O = \sum_{i=1}^m \langle [\![\boldsymbol{u}]\!]_i, \boldsymbol{n} \rangle \, \boldsymbol{\nu}_i \otimes \boldsymbol{t}_i, \tag{29}$$

where $[\![\boldsymbol{u}]\!]_i$ is the jump in displacement across the interface C_i at O; the other fields are also evaluated at O. It may appear from the form of $\boldsymbol{\gamma}_O$ in (29) that is not symmetric. However, this is not so. Indeed, given our regularity assumptions, $\boldsymbol{u}$ and its gradients are bounded almost everywhere in the vicinity of O implying that $\sum_{i=1}^m \langle [\![\boldsymbol{u}]\!]_i, \boldsymbol{n} \rangle = 0$. Noting that $\mathrm{skw}(\sum_{i=1}^m \langle [\![\boldsymbol{u}]\!]_i, \boldsymbol{n} \rangle \, \boldsymbol{\nu}_i \otimes \boldsymbol{t}_i) = \sum_{i=1}^m \langle [\![\boldsymbol{u}]\!]_i, \boldsymbol{n} \rangle (\boldsymbol{n} \times \boldsymbol{I})$ we have the desired result. For a single interface ($m = 1$), ending within S at O, $\sum_{i=1}^m \langle [\![\boldsymbol{u}]\!]_i, \boldsymbol{n} \rangle = 0$ implies that either the interface has no interior end point or $\langle [\![\boldsymbol{u}]\!], \boldsymbol{n} \rangle = 0$ at the interior end point of the interface.

Altogether there are six strain measures for a shell surface with imperfectly bonded interfaces, two bulk strains, three interfacial concentrations, and a point concentration, all given in terms of the piecewise smooth displacement field through the strain-displacement relations (21), defined on $S/\cup_{i=1}^m C_i$, (27), (28) on $\cup_{i=1}^m C_i$, and (29) at O. Notably, the strain concentrations $\boldsymbol{e}_C$, $\boldsymbol{\gamma}_1$, and $\boldsymbol{\gamma}_O$ do not appear for perfectly bonded interfaces and their junctions.

3.2 Compatibility conditions

In this section we derive the necessary and sufficient conditions on singular strain fields such that there exists a displacement field (of the required regularity) which satisfies the strain-displacement relations discussed above. We derive the distributional result first, corresponding to the strain-displacement relations (22). We will use the distributional result to obtain the strain compatibility conditions for shell surfaces with perfectly and imperfectly bonded interfaces.

Lemma 3.1. *Given* $\boldsymbol{E} \in \mathcal{D}'(S, \mathrm{Sym})$ *and* $\boldsymbol{\Lambda} \in \mathcal{D}'(S, \mathrm{Sym})$*, such that* $\boldsymbol{E}\boldsymbol{n} = \boldsymbol{\Lambda}\boldsymbol{n} = \boldsymbol{0}$*, there exists* $\boldsymbol{U} \in \mathcal{D}'(S, \mathbb{R}^3)$ *which satisfies* (22) *if and only if*

$$(\boldsymbol{E} - \boldsymbol{x} \times \boldsymbol{L})(\boldsymbol{\psi}) = 0 \text{ and } \boldsymbol{L}(\boldsymbol{\psi}) = 0, \tag{30}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathrm{Lin})$ *such that* $\mathrm{div}_S\, \boldsymbol{\psi} = \boldsymbol{0}$ *and* $\boldsymbol{\psi}\boldsymbol{n} = \boldsymbol{0}$*, where*

$$\boldsymbol{L} = \boldsymbol{n} \times \boldsymbol{\Lambda} - (\boldsymbol{n} \times \boldsymbol{E})\boldsymbol{b} - \boldsymbol{n} \otimes (\mathbb{P}\,\mathrm{Div}_S(\boldsymbol{E} \times \boldsymbol{n}))\,. \tag{31}$$

Proof. First, assuming that (30) hold, we establish the existence of a distributional displacement which satisfies (22). According to Lemma 2.3, $(30)_2$ implies the existence of $\boldsymbol{V} \in \mathcal{D}'(S, \mathbb{R}^3)$ such that $\nabla_S \boldsymbol{V} = \boldsymbol{L}$. Consequently $(30)_1$ implies the existence of $\boldsymbol{U}_1 \in \mathcal{D}'(S, \mathbb{R}^3)$ such that $\nabla_S \boldsymbol{U}_1 = \boldsymbol{E} - \boldsymbol{x} \times \nabla_S \boldsymbol{V}$. Let $\boldsymbol{U} = \boldsymbol{x} \times \boldsymbol{V} + \boldsymbol{U}_1$. Then, noting the identity $\nabla_S(\boldsymbol{x} \times \boldsymbol{V}) = \boldsymbol{x} \times \nabla_S \boldsymbol{V} - \boldsymbol{V} \times \mathbb{P}$, we obtain

$$\nabla_S \boldsymbol{U} = \boldsymbol{E} - \boldsymbol{V} \times \mathbb{P}. \tag{32}$$

It is then immediate that $\mathbb{P}\nabla_S \boldsymbol{U}\mathbb{P} = \boldsymbol{E} + \mathbb{P}(\boldsymbol{V} \times \mathbb{P})\mathbb{P}$, which leads to $\mathbb{P}\,\mathrm{sym}(\nabla_S \boldsymbol{U})\mathbb{P} = \boldsymbol{E}$. That $\mathbb{P}(\boldsymbol{V} \times \mathbb{P})\mathbb{P}$ is skew since $\langle \mathbb{P}(\boldsymbol{V} \times \mathbb{P})\mathbb{P}\boldsymbol{u}, \boldsymbol{u}\rangle = \langle \boldsymbol{V} \times (\mathbb{P}\boldsymbol{u}), \mathbb{P}\boldsymbol{u}\rangle = 0$ for any $\boldsymbol{u} \in \mathbb{R}^3$. To establish (22b) we begin by noting the identity $\boldsymbol{n}\nabla_S(\boldsymbol{V} \times \mathbb{P}) = \boldsymbol{n} \times \nabla_S \boldsymbol{V} + (\boldsymbol{n} \times \boldsymbol{V})\nabla_S \mathbb{P}$. Using it in conjunction with (32) we obtain

$$\mathbb{P}\boldsymbol{n}\nabla_S\nabla_S \boldsymbol{U}\mathbb{P} = \mathbb{P}\boldsymbol{n}\nabla_S \boldsymbol{E}\mathbb{P} - \mathbb{P}(\boldsymbol{n} \times \nabla_S \boldsymbol{V})\mathbb{P}, \tag{33}$$

where we have used $\mathbb{P}((\boldsymbol{n}\times\boldsymbol{V})\nabla_S\mathbb{P})\mathbb{P} = \boldsymbol{0}$. Recall that $\nabla_S \boldsymbol{V} = \boldsymbol{L}$, where $\boldsymbol{L}$ is defined in (31). The last term in (33) can therefore be expanded as $\mathbb{P}(\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\Lambda} - (\boldsymbol{n}\times\boldsymbol{E})\boldsymbol{b} - \boldsymbol{n}\otimes(\mathbb{P}\,\mathrm{Div}_S(\boldsymbol{E} \times \boldsymbol{n}))))\mathbb{P}$. In this expression $\mathbb{P}(\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\Lambda}))\mathbb{P} = -\boldsymbol{\Lambda}$, $\mathbb{P}(\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{E}\boldsymbol{b}))\mathbb{P} = -\boldsymbol{E}\boldsymbol{b}$, and $\mathbb{P}(\boldsymbol{n} \times (\boldsymbol{n} \otimes (\mathbb{P}\,\mathrm{Div}_S(\boldsymbol{E} \times \boldsymbol{n}))))\mathbb{P} = \boldsymbol{0}$. On the other hand, we note $\mathbb{P}\boldsymbol{n}\nabla_S \boldsymbol{E}\mathbb{P} = \boldsymbol{E}\boldsymbol{b}$. Substituting all of these into (33) reduces it to $\mathbb{P}\boldsymbol{n}\nabla_S\nabla_S \boldsymbol{U}\mathbb{P} = \boldsymbol{\Lambda}$.

In the second part of the proof, we assume the existence of a distributional displacement $\boldsymbol{U}$ which satisfies (22). We need to show that the strain fields thus defined satisfy the compatibility conditions (30). Let $\boldsymbol{V} \in \mathcal{D}'(S, \mathbb{R}^3)$ be such that $\boldsymbol{V} \times \mathbb{P} = \boldsymbol{E} - \nabla_S \boldsymbol{U}$. It follows immediately that $\boldsymbol{n} \times \nabla_S \boldsymbol{V} = \mathbb{P}\boldsymbol{n}\nabla_S \boldsymbol{E} - \boldsymbol{\Lambda}$, where we have used (22b) and $\mathbb{P}\boldsymbol{n}(\nabla_S(\boldsymbol{V}\times\mathbb{P})) = \boldsymbol{n}\times\nabla_S\boldsymbol{V}$. Furthermore, with $\boldsymbol{n}\times\boldsymbol{n}\times\nabla_S\boldsymbol{V} = -\mathbb{P}\nabla_S\boldsymbol{V}$ and $\boldsymbol{n}\times\mathbb{P}(\boldsymbol{n}\nabla_S\boldsymbol{E}) = \boldsymbol{n}\times\boldsymbol{E}\boldsymbol{b}$, we arrive at $\mathbb{P}\nabla_S\boldsymbol{V} = \boldsymbol{n}\times\boldsymbol{\Lambda} - \boldsymbol{n}\times\boldsymbol{E}\boldsymbol{b}$. On the other hand, using $\boldsymbol{E} = \boldsymbol{V}\times\mathbb{P} + \nabla_S\boldsymbol{U}$ and $\mathrm{Div}_S(\nabla_S\boldsymbol{U}\times\boldsymbol{n}) = \boldsymbol{0}$, we can obtain $(\nabla_S\boldsymbol{V})^T\boldsymbol{n} = -\mathbb{P}\,\mathrm{Div}_S(\boldsymbol{E}\times\boldsymbol{n})$. As a result of these calculations we therefore obtain

$$\nabla_S\boldsymbol{V} = \boldsymbol{n}\times\boldsymbol{\Lambda} - \boldsymbol{n}\times\boldsymbol{E}\boldsymbol{b} - \boldsymbol{n}\otimes\mathrm{Div}_S(\boldsymbol{E}\times\boldsymbol{n}) = \boldsymbol{L}, \tag{34}$$

where the second equality follows from (31). Finally, introducing $\boldsymbol{U}_1 = \boldsymbol{U} - \boldsymbol{x}\times\boldsymbol{V}$ yields

$$\nabla_S\boldsymbol{U}_1 = \boldsymbol{E} - \boldsymbol{x}\times\boldsymbol{L}. \tag{35}$$

According to Lemma 2.3, compatibility conditions (30) follow from (35) and the outer equality in (34), respectively. □

We will use Lemma 3.1 to derive local compatibility conditions on shell surfaces, both on and away from the singular interfaces, and also the global conditions which will follow from the topological nature of the shell surface. The derivation of the former rests on the following Lemma which provides a much simpler statement for the compatibility conditions whenever the shell surface is topologically equivalent to a disc.

Lemma 3.2. *If S is topologically equivalent to a disc then the compatibility equations* (30) *are reduced to*

$$\mathrm{Div}_S(\boldsymbol{L}\times\boldsymbol{n})(\boldsymbol{\phi}) = \boldsymbol{0}, \tag{36}$$

for all $\boldsymbol{\phi}\in\mathcal{D}(S,\mathbb{R}^3)$.

Proof. If S is topologically equivalent to a disc then, according to Lemma 2.4, $\mathrm{div}_S\,\boldsymbol{\psi} = \boldsymbol{0}$ and $\boldsymbol{\psi}\boldsymbol{n} = \boldsymbol{0}$, for $\boldsymbol{\psi}\in\mathcal{D}(S,\mathrm{Lin})$, is equivalent to the existence of $\boldsymbol{\phi}\in\mathcal{D}(S,\mathbb{R}^3)$ such that $\boldsymbol{\psi} = \nabla_S\boldsymbol{\phi}\times\boldsymbol{n}$. Recalling (7), this implies that conditions (30) are equivalent to

$$\mathrm{Div}_S((\boldsymbol{E}-\boldsymbol{x}\times\boldsymbol{L})\times\boldsymbol{n})(\boldsymbol{\phi}) = \boldsymbol{0} \text{ and } \mathrm{Div}_S(\boldsymbol{L}\times\boldsymbol{n})(\boldsymbol{\phi}) = \boldsymbol{0}, \tag{37}$$

for all $\boldsymbol{\phi}\in\mathcal{D}(S,\mathbb{R}^3)$. To complete the proof we will show that $(37)_1$ follows from $(37)_2$. For any $\boldsymbol{c}\in\mathrm{Lin}$ we associate $\mathrm{ax}(\boldsymbol{c})\in\mathbb{R}^3$ such that $\mathrm{ax}(\boldsymbol{u}\otimes\boldsymbol{v}) = \boldsymbol{u}\times\boldsymbol{v}$, where $\{\boldsymbol{u},\boldsymbol{v}\}\in\mathbb{R}^3$; hence, $\mathrm{ax}(\boldsymbol{c}) = \boldsymbol{0}$ for $\boldsymbol{c}\in\mathrm{Sym}$. If $\boldsymbol{c}$ is a surface tensor, i.e., $\mathbb{P}\boldsymbol{c}\mathbb{P} = \boldsymbol{c}$, we have $\mathrm{ax}(\boldsymbol{c}) = \langle\boldsymbol{c}\times\boldsymbol{n},\boldsymbol{I}\rangle\boldsymbol{n}$. Using the identity $\mathrm{Div}_S(\boldsymbol{x}\times\boldsymbol{A}) = \mathrm{ax}(\mathbb{P}\boldsymbol{A}^T) + \boldsymbol{x}\times\mathrm{Div}_S\,\boldsymbol{A}$ with $\boldsymbol{A} = (\boldsymbol{L}\times\boldsymbol{n})$, and $(37)_2$, we obtain $\mathrm{Div}_S(\boldsymbol{x}\times(\boldsymbol{L}\times\boldsymbol{n})) = \mathrm{ax}(\mathbb{P}(\boldsymbol{L}\times\boldsymbol{n})^T)$. The right hand side of this relation can be rewritten using (31), and noting that $\mathrm{ax}(\boldsymbol{n}\times\boldsymbol{\Lambda}\times\boldsymbol{n}) = \boldsymbol{0}$, $\mathrm{ax}(\boldsymbol{n}\times\boldsymbol{E}\boldsymbol{b}\times\boldsymbol{n})^T = -\langle\mathrm{Div}_S(\boldsymbol{E}\times\boldsymbol{n}),\boldsymbol{n}\rangle\boldsymbol{n}$, and $\mathrm{ax}(\mathbb{P}(\mathrm{Div}_S(\boldsymbol{E}\times\boldsymbol{n})\times\boldsymbol{n})\otimes\boldsymbol{n}) = -\mathbb{P}\,\mathrm{Div}_S(\boldsymbol{E}\times\boldsymbol{n})$, as $\mathrm{Div}_S(\boldsymbol{E}\times\boldsymbol{n})$. Consequently, $\mathrm{Div}_S(\boldsymbol{x}\times\boldsymbol{L}\times\boldsymbol{n}) = \mathrm{Div}_S(\boldsymbol{E}\times\boldsymbol{n})$. □

For a surface S with arbitrary topology the compatibility condition (36) will hold for all local neighbourhoods on S which are topologically equivalent to a disc. There will, however, be additional compatibility equations for each irreducible loop associated with the surface; these will follow directly from

(30). We will illustrate these ideas to derive local and global strain compatibility conditions for shell surfaces with perfectly and imperfectly bonded interfaces, for which the strain-displacement relations were discussed in the preceding section.

(i) Perfectly bonded interfaces. We consider an oriented surface S with m interfacial curves, C_i, and a point $O \in S$ such that $\partial C_i \subset \{O\} \cup \partial S$. Given a stretching strain field $\boldsymbol{e} \in \mathrm{Sym}$ and a bending strain field $\boldsymbol{\lambda} \in \mathrm{Sym}$, both piecewise smooth and uniformly bounded on S, but allowed to be discontinuous across C_i, and an interfacial monopole of bending strain concentration $\boldsymbol{\gamma} \in \mathrm{Sym}$, smooth and uniformly bounded over C_i, we seek necessary and sufficient compatibility conditions on these strain fields such that there exists a piecewise smooth and continuous displacement field $\boldsymbol{u} \in \mathbb{R}^3$, whose derivatives can be discontinuous across C_i, such that the strain-displacement relations (21), defined on S/C_i, and (25), on C_i, are satisfied. With the given strain fields, we construct distributions $\boldsymbol{E} \in \mathcal{B}(S, \mathrm{Sym})$ and $\boldsymbol{\Lambda} \in \mathcal{D}'(S, \mathrm{Sym})$ such that $\boldsymbol{E}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{e}, \boldsymbol{\psi} \rangle \, \mathrm{da}$ and $\boldsymbol{\Lambda}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{\lambda}, \boldsymbol{\psi} \rangle \, \mathrm{da} + \sum_{i=1}^m \int_{C_i} \langle \boldsymbol{\gamma}, \boldsymbol{\psi} \rangle \, \mathrm{dl}$, for all $\boldsymbol{\psi} \in \mathcal{D}(S, \mathrm{Lin})$. The distribution $\boldsymbol{L} \in \mathcal{D}'(S, \mathrm{Lin})$, defined in (31), can be evaluated as, for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathrm{Lin})$,

$$\boldsymbol{L}(\boldsymbol{\phi}) = \int_S \langle \boldsymbol{l}, \boldsymbol{\phi} \rangle \, \mathrm{da} + \sum_{i=1}^m \int_{C_i} \langle \boldsymbol{l}_C, \boldsymbol{\phi} \rangle \, \mathrm{da}, \text{ where} \tag{38a}$$

$$\boldsymbol{l} = \boldsymbol{n} \times \boldsymbol{\lambda} - (\boldsymbol{n} \times \boldsymbol{e})\boldsymbol{b} - \boldsymbol{n} \otimes \mathbb{P} \, \mathrm{div}_S(\boldsymbol{e} \times \boldsymbol{n}) \text{ and} \tag{38b}$$

$$\boldsymbol{l}_C = \boldsymbol{n} \times \boldsymbol{\gamma} - \boldsymbol{n} \otimes [\![\boldsymbol{e}]\!]\boldsymbol{t}. \tag{38c}$$

The compatibility conditions require that $\boldsymbol{E}\boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{\Lambda}\boldsymbol{n} = \boldsymbol{0}$ or equivalently $\boldsymbol{e}\boldsymbol{n} = \boldsymbol{0}$, $\boldsymbol{\lambda}\boldsymbol{n} = \boldsymbol{0}$, and $\boldsymbol{\gamma}\boldsymbol{n} = \boldsymbol{0}$. The local compatibility conditions can be obtained using (36), which on substituting (38) and recalling (16) yields

$$\mathrm{div}_S(\boldsymbol{l} \times \boldsymbol{n}) = \boldsymbol{0} \text{ in } S/\cup_{i=1}^m C_i, \tag{39a}$$

$$\boldsymbol{n} \times (\boldsymbol{\gamma}\boldsymbol{t}) + \boldsymbol{n} \, \langle [\![\boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{t} \rangle = \boldsymbol{0} \text{ on } C_i, \text{ and} \tag{39b}$$

$$\begin{aligned} &\frac{d}{ds}\Big(\boldsymbol{n} \times \boldsymbol{\gamma} \times \boldsymbol{n} + \boldsymbol{n} \otimes ([\![\boldsymbol{e}]\!]\boldsymbol{t} \times \boldsymbol{n})\Big)\boldsymbol{t} + k_g(\boldsymbol{n} \times (\boldsymbol{\gamma}\boldsymbol{t}) + \boldsymbol{n} \, \langle [\![\boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{t} \rangle) + \\ &\qquad \boldsymbol{n} \times ([\![\boldsymbol{\lambda}]\!]\boldsymbol{t} - [\![\boldsymbol{e}]\!]\boldsymbol{b}\boldsymbol{t}) - \boldsymbol{n}\langle [\![\mathrm{div}_S(\boldsymbol{e} \times \boldsymbol{n})]\!], \boldsymbol{t} \rangle = \boldsymbol{0} \text{ on } C_i. \end{aligned} \tag{39c}$$

We can simplify these equations. Equation (39b) is equivalent to

$$\boldsymbol{\gamma}\boldsymbol{t} = \boldsymbol{0} \text{ and } \langle [\![\boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{t} \rangle = 0, \tag{40}$$

which, when substituted into (39c), reduces it to

$$2 \, \langle [\![\nabla_S \boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{\nu} \otimes \boldsymbol{t} \rangle - \langle [\![\nabla_S \boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle + k_n \, \langle \boldsymbol{\gamma}, \boldsymbol{\nu} \otimes \boldsymbol{\nu} \rangle + k_g \, \langle [\![\boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{t} \rangle = 0 \text{ and} \tag{41a}$$

$$[\![\boldsymbol{\lambda}]\!]\boldsymbol{t} + \frac{d}{ds}(\boldsymbol{\gamma}\boldsymbol{\nu}) - 2k_n \langle [\![\boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle \boldsymbol{t} - \tau_g \langle [\![\boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle \boldsymbol{\nu} = \boldsymbol{0}. \tag{41b}$$

The compatibility conditions (39) are therefore equivalent to (39a), (40), and (41). The junction compatibility conditions at O are obtained following the derivation of $(20\text{d})_1$ as

$$\sum_{i=1}^m \boldsymbol{\gamma}_i \boldsymbol{\nu}_i = \boldsymbol{0} \text{ and } \sum_{i=1}^m \langle [\![\boldsymbol{e}]\!]_i, \boldsymbol{t}_i \otimes \boldsymbol{\nu}_i \rangle = 0, \tag{42}$$

where $(\cdot)_i$ indicates the limiting value of the field on C_i at it approaches O. According to $(40)_1$ and $(42)_1$ it is immediate that a single interface ($m = 1$) cannot end within the domain. To permit a single fold to end within the domain we will have to relax the regularity of the bulk fields and allow them to become unbounded as they approach O [10]. Multiple folds can of course end at their junction point as long as the compatibility conditions are satisfied.

Additional compatibility conditions are needed if the shell is multiply connected. These conditions are non-local (global) and reflect the restrictions due to the topological character of the shell surface. The global conditions follow directly from (30). Let $\boldsymbol{M} = \boldsymbol{E} - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{L}$, where $\boldsymbol{x}_0$ is an arbitrary fixed position. Using the distributional forms of $\boldsymbol{E}$ and $\boldsymbol{L}$ from the above discussion, we can write

$$\boldsymbol{M}(\boldsymbol{\phi}) = \int_S \langle \boldsymbol{m}, \boldsymbol{\phi} \rangle \,\mathrm{da} + \sum_{i=1}^{m} \int_{C_i} \langle \boldsymbol{m}_C, \boldsymbol{\phi} \rangle \,\mathrm{dl}, \text{ where} \tag{43a}$$

$$\boldsymbol{m} = \boldsymbol{e} - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l} \text{ and} \tag{43b}$$

$$\boldsymbol{m}_C = -(\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_C, \tag{43c}$$

for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathrm{Lin})$. The two relations in (30) can be written as a single equation $\boldsymbol{M}(\boldsymbol{\phi}) = \boldsymbol{0}$, which on using (43), and following the derivation leading to (20e), yields

$$\int_L (\boldsymbol{et} - (\boldsymbol{x} - \boldsymbol{x}_0) \times (\boldsymbol{lt})) \,\mathrm{dl} - \sum_{i=1}^{m} (\boldsymbol{x} - \boldsymbol{x}_0) \times (\boldsymbol{l}_C \boldsymbol{\nu})|_{C_i \cap L} = \boldsymbol{0}, \tag{44}$$

where $\boldsymbol{x}_0$ is an arbitrary fixed position, for all independent irreducible loops L on S. Altogether, the required compatibility equations for singular strain fields on a shell surface with perfectly bonded interfaces are given by (39a), (40), (41), (42), and (44). The desired regularity of the displacement field $\boldsymbol{u}$ follows from the results in Section 2.6.

Remark 3.1. (Folds on shells with vanishing bulk strains) We can use the compatibility equations to investigate the nature of a fold on S such that there are no bulk strains (i.e., $\boldsymbol{e} = \boldsymbol{\lambda} = \boldsymbol{0}$). In other words, we want to know the ways in which a shell surface can be folded without generating any bulk strains. The only non-trivial strain field is therefore the interfacial bending strain $\boldsymbol{\gamma} \in \mathrm{Sym}$. According to $(40)_1$, we can write $\boldsymbol{\gamma}$ in terms of a smooth scalar field a on C such that $\boldsymbol{\gamma} = a\boldsymbol{\nu} \otimes \boldsymbol{\nu}$. Furthermore, $(40)_2$ is trivially satisfied whereas (41) reduces to $ak_n = 0$ and $(\mathrm{d}a/\mathrm{ds})\boldsymbol{\nu} - ak_g\boldsymbol{t} + a\tau_g\boldsymbol{n} = \boldsymbol{0}$. Accordingly, for a non-trivial fold ($a \neq 0$) to exist, $\mathrm{d}a/\mathrm{ds} = 0$, ans $k_n = k_g = \tau_g = 0$. The fold is therefore necessarily a straight line with constant fold strength. A curved fold interface or a non-uniform fold strength would therefore necessarily induce non-zero strains in the bulk.

(ii) Imperfectly bonded interfaces. We consider an oriented surface S with m interfacial curves, C_i, and a point $O \in S$ such that $\partial C_i \subset \{O\} \cup \partial S$. Given a stretching strain field $\boldsymbol{e} \in \mathrm{Sym}$ and a bending strain field $\boldsymbol{\lambda} \in \mathrm{Sym}$, both piecewise smooth and uniformly bounded over S, but allowed to be discontinuous across C_i, an interfacial concentration of stretching strain $\boldsymbol{e}_C \in \mathrm{Sym}$, an interfacial monopole of bending strain concentration $\boldsymbol{\gamma} \in \mathrm{Sym}$, an interfacial dipole of bending strain concentration $\boldsymbol{\gamma}_1 \in \mathrm{Sym}$, all smooth and uniformly bounded over C_i, and a point concentration $\boldsymbol{\gamma}_O \in \mathrm{Sym}$ in bending strain at O, we seek

necessary and sufficient compatibility conditions on these strain fields such that there exists a piecewise smooth displacement field $\boldsymbol{u} \in \mathbb{R}^3$, allowed to be discontinuous across C_i, such that the strain-displacement relations (21), defined on S/C, and (27), (28) on C, are satisfied. With the given strain fields, we construct distributions $\boldsymbol{E} \in \mathcal{D}'(S, \text{Sym})$ and $\boldsymbol{\Lambda} \in \mathcal{D}'(S, \text{Sym})$ such that $\boldsymbol{E}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{e}, \boldsymbol{\psi} \rangle \, \text{da} + \sum_{i=1}^m \int_{C_i} \langle \boldsymbol{e}_C, \boldsymbol{\psi} \rangle \, \text{dl}$ and $\boldsymbol{\Lambda}(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{\lambda}, \boldsymbol{\psi} \rangle \, \text{da} + \sum_{i=1}^m \int_{C_i} \langle \boldsymbol{\gamma}, \boldsymbol{\psi} \rangle \, \text{dl} + \sum_{i=1}^m \int_{C_i} \langle \boldsymbol{\gamma}_1, \partial \boldsymbol{\psi}/\partial \nu \rangle \, \text{dl} + \langle \boldsymbol{\gamma}_O, \boldsymbol{\psi}(O) \rangle$, for all $\boldsymbol{\psi} \in \mathcal{D}(S, \text{Lin})$. The distribution $\boldsymbol{L} \in \mathcal{D}'(S, \text{Lin})$, defined in (31), can be evaluated as, for all $\boldsymbol{\phi} \in \mathcal{D}(S, \text{Lin})$,

$$\boldsymbol{L}(\boldsymbol{\phi}) = \int_S \langle \boldsymbol{l}, \boldsymbol{\phi} \rangle \, \text{da} + \int_C \langle \boldsymbol{l}_C, \boldsymbol{\phi} \rangle \, \text{dl} + \int_C \left\langle \boldsymbol{l}_F, \frac{\partial \boldsymbol{\phi}}{\partial \nu} \right\rangle \text{dl} + \langle \boldsymbol{l}_O, \boldsymbol{\phi}(O) \rangle, \text{ where} \tag{45a}$$

$$\boldsymbol{l} = \boldsymbol{n} \times \boldsymbol{\lambda} - (\boldsymbol{n} \times \boldsymbol{e})\boldsymbol{b} - \boldsymbol{n} \otimes (\mathbb{P} \, \text{div}_S(\boldsymbol{e} \times \boldsymbol{n})), \tag{45b}$$

$$\begin{aligned} \boldsymbol{l}_C = \boldsymbol{n} \times \boldsymbol{\gamma} + \frac{\partial \boldsymbol{n}}{\partial \nu} \times \boldsymbol{\gamma}_1 - \frac{\partial \boldsymbol{n}}{\partial \nu} \otimes (\boldsymbol{e}_C \boldsymbol{t}) + \boldsymbol{n} \otimes \boldsymbol{n} \left\langle \frac{\partial \boldsymbol{n}}{\partial \nu}, (\boldsymbol{e}_C \boldsymbol{t}) \right\rangle - \boldsymbol{n} \otimes ([\![\boldsymbol{e}]\!] \boldsymbol{t}) - \\ (\boldsymbol{n} \times \boldsymbol{e}_C)\boldsymbol{b} - \boldsymbol{n} \otimes \mathbb{P} \left(\frac{d\boldsymbol{e}_C}{ds} \boldsymbol{\nu} \right) + k_g \boldsymbol{n} \otimes (\boldsymbol{e}_C \boldsymbol{t}), \end{aligned} \tag{45c}$$

$$\boldsymbol{l}_F = \boldsymbol{n} \times \boldsymbol{\gamma}_1 - \boldsymbol{n} \otimes (\boldsymbol{e}_C \boldsymbol{t}), \text{ and} \tag{45d}$$

$$\boldsymbol{l}_O = \boldsymbol{n} \times \boldsymbol{\gamma}_O + \sum_{i=1}^m \boldsymbol{n} \otimes \boldsymbol{e}_{C_i} \boldsymbol{\nu}_i. \tag{45e}$$

The compatibility conditions require that $\boldsymbol{E}\boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{\Lambda}\boldsymbol{n} = \boldsymbol{0}$ or equivalently $\boldsymbol{e}\boldsymbol{n} = \boldsymbol{0}$, $\boldsymbol{\lambda}\boldsymbol{n} = \boldsymbol{0}$, $\boldsymbol{e}_C \boldsymbol{n} = \boldsymbol{0}$, $\boldsymbol{\gamma}_1 \boldsymbol{n} = \boldsymbol{0}$, $\boldsymbol{\gamma}_O \boldsymbol{n} = \boldsymbol{0}$, and

$$\boldsymbol{\gamma}\boldsymbol{n} - \tau_g \boldsymbol{\gamma}_1 \boldsymbol{t} - (k - k_n)\boldsymbol{\gamma}_1 \boldsymbol{\nu} = \boldsymbol{0}. \tag{46}$$

The local compatibility conditions can be obtained using (36), on substituting (45) and recalling (16), as

$$\text{div}_S(\boldsymbol{l} \times \boldsymbol{n}) = \boldsymbol{0} \text{ in } S/\cup_{i=1}^m C_i, \tag{47a}$$

$$\boldsymbol{n} \times (\boldsymbol{\gamma}_1 \boldsymbol{t}) - \boldsymbol{n} \langle \boldsymbol{e}_C, \boldsymbol{t} \otimes \boldsymbol{t} \rangle = \boldsymbol{0} \text{ on } C_i, \tag{47b}$$

$$\begin{aligned} -\boldsymbol{n} \left(2 \left\langle \frac{d\boldsymbol{e}_C}{ds}, \boldsymbol{\nu} \otimes \boldsymbol{t} \right\rangle + k_g \langle \boldsymbol{e}_C, \boldsymbol{\nu} \otimes \boldsymbol{\nu} \rangle + k_n \langle \boldsymbol{\gamma}_1, \boldsymbol{\nu} \otimes \boldsymbol{\nu} \rangle + \langle [\![\boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{t} \rangle \right) + \\ \boldsymbol{n} \times (\boldsymbol{\gamma} \boldsymbol{t}) - (\boldsymbol{n} \times \boldsymbol{e}_C)(\boldsymbol{b}\boldsymbol{t}) + \boldsymbol{n} \times \left(\frac{d\boldsymbol{\gamma}_1}{ds} \boldsymbol{\nu} \right) - \frac{\partial \boldsymbol{n}}{\partial s} \langle \boldsymbol{e}_C, \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle = \boldsymbol{0} \text{ on } C_i, \text{ and} \end{aligned} \tag{47c}$$

$$\begin{aligned} \boldsymbol{n} \times ([\![\boldsymbol{\lambda}]\!] \boldsymbol{t} - [\![\boldsymbol{e}]\!] \boldsymbol{b}\boldsymbol{t}) - \boldsymbol{n} \langle [\![\text{div}_S(\boldsymbol{e} \times \boldsymbol{n})]\!], \boldsymbol{t} \rangle + \frac{dk_g}{ds} (\boldsymbol{n} \times (\boldsymbol{\gamma}_1 \boldsymbol{\nu})) - \frac{dk_g}{ds} \langle \boldsymbol{e}_C, \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle \boldsymbol{n} \\ + k_g \boldsymbol{n} \times \left(\frac{d\boldsymbol{\gamma}_1}{ds} \boldsymbol{\nu} \right) - k_g \frac{d\boldsymbol{n}}{ds} \langle \boldsymbol{e}_C, \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle + k_g \frac{d\boldsymbol{n}}{ds} \times (\boldsymbol{\gamma}_1 \boldsymbol{\nu}) - k_g \boldsymbol{n} \left\langle \frac{d(\boldsymbol{e}_C \boldsymbol{t})}{ds}, \boldsymbol{\nu} \right\rangle + \frac{d}{ds} (\boldsymbol{n} \times \boldsymbol{\gamma} \boldsymbol{\nu}) + \\ + \frac{d}{ds} \left(\frac{\partial \boldsymbol{n}}{\partial \nu} \times \boldsymbol{\gamma}_1 \boldsymbol{\nu} - \frac{\partial \boldsymbol{n}}{\partial \nu} \langle \boldsymbol{e}_C, \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle - \boldsymbol{n} \langle [\![\boldsymbol{e}]\!], \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle - \boldsymbol{n} \times \boldsymbol{e}_C (\boldsymbol{b}\boldsymbol{\nu}) - \right. \\ \left. \boldsymbol{n} \left\langle \frac{d\boldsymbol{e}_C}{ds}, \boldsymbol{\nu} \otimes \boldsymbol{\nu} \right\rangle + k_g \boldsymbol{n} \langle \boldsymbol{e}_C, \boldsymbol{t} \otimes \boldsymbol{\nu} \rangle \right) = \boldsymbol{0} \text{ on } C_i. \end{aligned} \tag{47d}$$

The junction compatibility conditions at O are obtained following the derivation of (20d) as

$$\boldsymbol{\gamma}_0 - \frac{1}{2}\sum_{i=1}^{m}\langle \boldsymbol{\gamma}_{1i}, \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i\rangle\,(\boldsymbol{\nu}_i \otimes \boldsymbol{t}_i + \boldsymbol{t}_i \otimes \boldsymbol{\nu}_i) = \boldsymbol{0}, \tag{48a}$$

$$\sum_{i=1}^{m}\langle \boldsymbol{\gamma}_{1i}, \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i\rangle = 0, \tag{48b}$$

$$\sum_{i=1}^{m} 2\langle \boldsymbol{e}_{ci}, \boldsymbol{t}_i \otimes \boldsymbol{\nu}_i\rangle \boldsymbol{t}_i + \sum_{i=1}^{m}\langle \boldsymbol{e}_{ci}, \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i\rangle \boldsymbol{\nu}_i = \boldsymbol{0}, \tag{48c}$$

$$\sum_{i=1}^{m}\left\{\left\langle \frac{d\boldsymbol{e}_{ci}}{ds}, \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i\right\rangle + \langle [\![\boldsymbol{e}_i]\!], \boldsymbol{t}_i \otimes \boldsymbol{\nu}_i\rangle + \tau_g\,\langle \boldsymbol{\gamma}_{1i}, \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i\rangle\right\} = 0, \text{ and} \tag{48d}$$

$$\sum_{i=1}^{m} \boldsymbol{n} \times \left\{\boldsymbol{\gamma}_i \boldsymbol{\nu}_i - (2\tau_g\langle \boldsymbol{e}_{Ci}, \boldsymbol{t}_i \otimes \boldsymbol{\nu}_i\rangle \boldsymbol{t}_i + \langle (k-k_n)\boldsymbol{e}_{Ci}, \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i\rangle \boldsymbol{\nu}_i) + k_{g_i}\boldsymbol{\gamma}_{1i}\boldsymbol{\nu}_i\right\} = \boldsymbol{0}. \tag{48e}$$

According to $(47\text{b})_1$ and (48b) it is immediate that a single interface ($m = 1$) cannot end within the domain. To permit a single tear to end within the domain we will have to relax the regularity of the bulk fields and allow them to become unbounded as they approach O [10]. The global conditions follow directly from (30). Let $\boldsymbol{M} = \boldsymbol{E} - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{L}$, where $\boldsymbol{x}_0$ is an arbitrary fixed position. Using the distributional forms of $\boldsymbol{E}$ and $\boldsymbol{L}$ given above, we can write

$$\boldsymbol{M}(\boldsymbol{\phi}) = \int_S \langle \boldsymbol{m}_B, \boldsymbol{\phi}\rangle\,\mathrm{da} + \sum_{i=1}^{m}\int_{C_i}\langle \boldsymbol{m}_C, \boldsymbol{\phi}\rangle\,\mathrm{dl} + \sum_{i=1}^{m}\int_{C_i}\langle \boldsymbol{m}_F, \frac{\partial \boldsymbol{\phi}}{\partial \nu}\rangle\,\mathrm{dl} + \langle \boldsymbol{m}_O, \boldsymbol{\phi}(O)\rangle, \text{ where} \tag{49a}$$

$$\boldsymbol{m} = \boldsymbol{e} - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_B, \tag{49b}$$

$$\boldsymbol{m}_C = \boldsymbol{e}_C - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_C - \boldsymbol{\nu} \times \boldsymbol{l}_F, \tag{49c}$$

$$\boldsymbol{m}_F = -(\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_F, \text{ and} \tag{49d}$$

$$\boldsymbol{m}_O = -(\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_O, \tag{49e}$$

for all $\boldsymbol{\phi} \in \mathcal{D}(S, \mathrm{Lin})$. The two relations in (30) can be written as a single equation $\boldsymbol{M}(\boldsymbol{\phi}) = \boldsymbol{0}$, which on using (49), and following the derivation leading to (20e), yields

$$\int_L (\boldsymbol{e}\boldsymbol{t} - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_B \boldsymbol{t})\,\mathrm{dl} + \sum_{i=1}^{m}\left\{\boldsymbol{e}_{Ci}\boldsymbol{\nu}_i - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_{Ci}\boldsymbol{\nu}_i - \boldsymbol{\nu}_i \times \boldsymbol{l}_{Fi}\boldsymbol{\nu}_i - k_{g_i}(\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_{Fi}\boldsymbol{\nu}_i\right\}|_{C_i \cap L} = \boldsymbol{0}, \tag{50}$$

for all independent irreducible loops L on S. Altogether, the required compatibility equations for singular strain fields on a shell surface with imperfectly bonded interfaces are given by (47), (48), and (50). The desired regularity of the displacement field $\boldsymbol{u}$ follows from the results in Section 2.6. We note that $\boldsymbol{m}_O$ does not contribute to the global compatibility condition since we can always consider a loop L such that $O \notin L$.

It is a minor exercise to show that the junction equations (48b)-(48e) are equivalent to requiring

$$\lim_{\epsilon \to 0}\Bigg(\int_{L_\epsilon} (\boldsymbol{e}\boldsymbol{t} - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_B \boldsymbol{t})\,\mathrm{dl} + \\ \sum_{i=1}^{m}\left\{\boldsymbol{e}_{Ci}\boldsymbol{\nu}_i - (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_{Ci}\boldsymbol{\nu}_i - \boldsymbol{\nu}_i \times \boldsymbol{l}_{Fi}\boldsymbol{\nu}_i - k_{g_i}(\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{l}_{Fi}\boldsymbol{\nu}_i\right\}|_{C_i \cap L_\epsilon}\Bigg) = \boldsymbol{0}, \tag{51}$$

where L_ϵ is a loop of length ϵ around O. In fact, the line integral in (51) vanishes identically due to the boundedness of strain fields. Notice the similarity of the form of (50) and (51). Writing junction conditions junction equations (48b)-(48e) equivalently in the form (51) emphasises the topological significance of these equations. Let $B_\epsilon \subset S$ be a topological disc of radius ϵ and center at O; hence $\partial B_\epsilon = L_\epsilon$. Consider S to be topologically a disc (a simply connected domain) and let $S_\epsilon = S - B_\epsilon$ be the punctured domain (now multiply connected) by cutting out B_ϵ from S. While the compatibility conditions for S include (47) and (48), the compatibility conditions for S_ϵ consist of (47) and (50) with the latter written for $L = L_\epsilon$. in the limit $\epsilon \to 0$ all the junction conditions, except (48a), are recovered. Consider now S to be topologically a sphere (simply connected but non-contractible) and let $S_\epsilon = S - B_\epsilon$ be the punctured domain (simply connected topologically a disc). The compatibility conditions for S include (47) and (48) whereas the compatibility conditions for S_ϵ consist of only (47) (we are focussing on O; there may be other junctions on S where the junction compatibility conditions will hold). The limiting case of S_ϵ, $\epsilon \to 0$, suggests that the junctions conditions (48b)-(48e) for S are trivially satisfied. Indeed, their alternate form (51) is identically satisfied due to the bulk compatibility (47) in $S - B_\epsilon$. Consequently, if there are k junctions on a sphere S, the junction conditions (48b)-(48e) have to be imposed on only $k-1$ junctions. The condition (48a), however, needs to be satisfied at all the junction points.

It is clear from the preceding paragraph that the junction condition (48a) is different from other junction conditions, most significantly in its non-topological nature. To understand the role of (48a), we consider all the compatibility conditions to be satisfied except (48a). There then exists a displacement field (of required regularity) within $S - B_\epsilon$ but it is only through (48a) that, as $\epsilon \to 0$, it can be extended across ∂B_ϵ to the complete surface S. This also emphasises the local nature of the condition. We note that given the regularity of strains in the vicinity of O, (48a) is sufficient for compatibility across ∂B_ϵ. For more singular strain fields, we expect further local conditions to ensure compatibility in the vicinity of O.

Remark 3.2. (Fold and fold dipoles in the absence of bulk fields) Consider a simply connected surface S with an interfacial curve $C \subset S$ such that $\partial C - \partial S = \emptyset$. The latter condition requires C to end either at the boundary of S or on itself. Assume both the bending and stretching strains to be supported only at the interface C, i.e., $\boldsymbol{e} = \boldsymbol{\lambda} = \boldsymbol{0}$. We obtain the implications of compatibility relations for such strain fields. According to (47b) $(\boldsymbol{I} - \boldsymbol{t} \otimes \boldsymbol{t})\boldsymbol{\gamma}_1 \boldsymbol{t} = \boldsymbol{0}$ and $\langle \boldsymbol{e}_c, \boldsymbol{t} \otimes \boldsymbol{t} \rangle = 0$, which implies that there exists a smooth scalar field $a : C \to \mathbb{R}$ satisfying $\boldsymbol{\gamma}_1 = a\boldsymbol{\nu} \otimes \boldsymbol{\nu}$ and a smooth vector field $\boldsymbol{c} : C \to \mathbb{R}^3$ such that $\langle \boldsymbol{c}, \boldsymbol{n} \rangle = 0$ and $\boldsymbol{e}_c = \frac{1}{2}(\boldsymbol{c} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{c})$. Substituting these in (46) and the tangential projection of (47c) yields

$$\boldsymbol{\gamma n} = (k - k_n)a\boldsymbol{\nu} \text{ and } \boldsymbol{n} \times \boldsymbol{\gamma t} = \left(\left\langle \frac{d\boldsymbol{d}}{ds}, \boldsymbol{n} \right\rangle \right) \boldsymbol{\nu} + k_g a \boldsymbol{t}, \tag{52}$$

respectively, where $\boldsymbol{d} = -\boldsymbol{c} + a\boldsymbol{n}$. As a result, introducing a smooth scalar field $\theta : C \to \mathbb{R}$, we can write

$$\boldsymbol{\gamma} = k_g a \boldsymbol{t} \otimes \boldsymbol{t} - \left(\left\langle \frac{d\boldsymbol{d}}{ds}, \boldsymbol{n} \right\rangle \right) (\boldsymbol{t} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{t}) + \theta \boldsymbol{\nu} \otimes \boldsymbol{\nu} + (k - k_n)a(\boldsymbol{n} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{n}). \tag{53}$$

On the other hand, projecting (47c) along the normal direction yields $\langle d\boldsymbol{d}/ds, \boldsymbol{t} \rangle = 0$ implying existence of a smooth vector field $\boldsymbol{f} : C \to \mathbb{R}^3$ satisfying $\langle \boldsymbol{f}, \boldsymbol{t} \rangle = 0$ and $d\boldsymbol{d}/ds = \boldsymbol{t} \times \boldsymbol{f}$. Substituting these representations in (47d) we have

$$\frac{d((\theta - \tau_g \langle \boldsymbol{c}, \boldsymbol{t}\rangle - (k - k_n)\langle \boldsymbol{c}, \boldsymbol{\nu}\rangle + k_g a)\boldsymbol{t})}{ds} + \frac{d\boldsymbol{f}}{ds} = \boldsymbol{0}. \tag{54}$$

It immediately follows that the vector field $\boldsymbol{g} = (\theta - \tau_g\langle \boldsymbol{c}, \boldsymbol{t}\rangle - (k-k_n)\langle \boldsymbol{c}, \boldsymbol{\nu}\rangle + k_g a)\boldsymbol{t} + \boldsymbol{f}$ satisfies $d\boldsymbol{d}/ds = \boldsymbol{t}\times\boldsymbol{g}$ and $d\boldsymbol{g}/ds = \boldsymbol{0}$. Hence,

$$\boldsymbol{d} = \boldsymbol{a} + \boldsymbol{x} \times \boldsymbol{g}, \tag{55}$$

where $\boldsymbol{a}, \boldsymbol{g} \in \mathbb{R}^3$ are both constant. In fact, $\boldsymbol{a}$ represents a translational jump across the interface and $\boldsymbol{g}$ represents a rotational jump across the interface C.

In case we would have considered the curve C to be such that it has an end point O within S, then the junction compatibility (48), with $m = 1$, would yield that $\boldsymbol{d}$ is necessarily zero (and hence an interface as considered above will not be possible). Indeed, (48b) and (48c) imply that the translational jump projected both in the normal and the tangential direction to the surface must be 0. Similarly, (48d) is satisfied if the normal component of the rotational jump $\boldsymbol{g}$ is 0 and (48d) is satisfied if the surface projection of $\boldsymbol{g}$ is $\boldsymbol{0}$. Such interfaces, ending within the domain and carrying jumps in displacements as given by (55), do exist when there are topological defects (dislocations, disclinations) at O. Such nontrivial interfaces are not supported in our framework due to boundedness assumptions on strain fields. We would be required to allow the strain fields to become unbounded as they approach O.

Remark 3.3. (Folds and tears in plates) The compatibility results are now reduced for the case of plates, i.e., when the reference surface S is flat and hence the second fundamental form $\boldsymbol{b}$ vanishes identically (the normal field $\boldsymbol{n}$ is uniform). Equivalently, using (3), we can impose flatness of S by substituting $k_n = 0$, $\tau_g = 0$, and $k = 0$. We discuss only the imperfectly bonded case with a single interfacial curve C such that $\partial C - \partial S = \emptyset$. The strain displacement equations are as given by (21), defined on S/C, and (27), (28) on C, with (28a) now reduced to

$$\boldsymbol{\gamma} = -\left\langle \boldsymbol{n}, \left[\!\!\left[\frac{\partial \boldsymbol{u}}{\partial s}\right]\!\!\right] \right\rangle (\boldsymbol{t}\otimes\boldsymbol{\nu} + \boldsymbol{\nu}\otimes\boldsymbol{t}) - \left\langle \boldsymbol{n}, \left[\!\!\left[\frac{\partial \boldsymbol{u}}{\partial \nu}\right]\!\!\right] \right\rangle (\boldsymbol{\nu}\otimes\boldsymbol{\nu}) + k_g \langle [\![\boldsymbol{u}]\!], \boldsymbol{n}\rangle (\boldsymbol{t}\otimes\boldsymbol{t} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}) \tag{56}$$

We seek necessary and sufficient conditions on uniformly bounded symmetric strain fields for there to exist a piecewise smooth displacement field, possible discontinuous on C. The compatibility conditions require that $\boldsymbol{E}\boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{\Lambda}\boldsymbol{n} = \boldsymbol{0}$ which for the plate are equivalent to $\boldsymbol{e}\boldsymbol{n} = \boldsymbol{\lambda}\boldsymbol{n} = \boldsymbol{\gamma}_1\boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{\gamma}\boldsymbol{n} = \boldsymbol{0}$. The latter is a simplified form of (46). The distributional compatibility condition (36), with $\boldsymbol{L}$ given in (31), reduces to a decoupled system of conditions for stretching and bending strain distributions representing in-plane and normal compatibility conditions. This is due to the uniformity of $\boldsymbol{n}$ over S. Using (36) and (31) we obtain

$$\mathrm{Div}_S(\mathrm{Div}_S(\boldsymbol{E}\times\boldsymbol{n})\times\boldsymbol{n}) = 0 \text{ and } \mathrm{Div}_S(\boldsymbol{\Lambda}\times\boldsymbol{n}) = \boldsymbol{0}. \tag{57}$$

These can be localised to write the point wise compatibility conditions in S as

$$\text{div}_S(\text{div}_S(\boldsymbol{e}\times\boldsymbol{n})\times\boldsymbol{n}) = 0 \text{ and } \text{div}_S(\boldsymbol{\lambda}\times\boldsymbol{n}) = \boldsymbol{0} \text{ on } S/C, \tag{58a}$$

$$2\left\langle \frac{d\boldsymbol{e}_C}{ds}, \boldsymbol{\nu}\otimes\boldsymbol{t}\right\rangle + k_g\left\langle \boldsymbol{e}_C, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\right\rangle + \langle [\![\boldsymbol{e}]\!], \boldsymbol{t}\otimes\boldsymbol{t}\rangle = 0 \text{ on } C, \tag{58b}$$

$$\langle [\![\text{div}_S(\boldsymbol{e}\times\boldsymbol{n})]\!], \boldsymbol{t}\rangle - k_g\frac{d}{ds}\left\langle \boldsymbol{e}_C, \boldsymbol{t}\otimes\boldsymbol{\nu}\right\rangle + k_g\left\langle \frac{d(\boldsymbol{e}_C\boldsymbol{t})}{ds}, \boldsymbol{\nu}\right\rangle + \frac{d}{ds}\left(\langle [\![\boldsymbol{e}]\!], \boldsymbol{t}\otimes\boldsymbol{\nu}\rangle + \left\langle \frac{d\boldsymbol{e}_C}{ds}, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\right\rangle\right) = 0 \text{ on } C, \tag{58c}$$

$$\langle \boldsymbol{e}_C, \boldsymbol{t}\otimes\boldsymbol{t}\rangle = 0 \text{ on } C, \tag{58d}$$

$$\boldsymbol{\gamma}\boldsymbol{t} + \frac{d\boldsymbol{\gamma}_1}{ds}\boldsymbol{\nu} = \boldsymbol{0},\ \boldsymbol{\gamma}_1\boldsymbol{t} = \boldsymbol{0} \text{ on } C, \text{ and} \tag{58e}$$

$$[\![\boldsymbol{\lambda}]\!]\boldsymbol{t} + \frac{dk_g}{ds}(\boldsymbol{\gamma}_1\boldsymbol{\nu}) + k_g\left(\frac{d\boldsymbol{\gamma}_1}{ds}\boldsymbol{\nu}\right) + \frac{d}{ds}(\boldsymbol{\gamma}\boldsymbol{\nu}) = \boldsymbol{0} \text{ on } C. \tag{58f}$$

These equations should be compared with their analogous counterparts in our previous work on plane elasticity and von Kármán plate theory [8, 10]. If the plate domain has holes then suitable topological conditions of the form (50) have to be additionally satisfied.

4 An incompatibility problem

We illustrate an application of our framework. Let S be a regular, oriented, and simply connected shell surface with a single imperfectly bonded interfacial curve C such that $\partial C - \partial S = \emptyset$. We consider a situation where the strain concentrations are purely plastic in nature, i.e., they do not contribute to stored energy of the shell. The bulk strain, on the other hand, are considered to be purely elastic and hence have a corresponding stress and moment tensor associated with them through a constitutive model. The plastic strains can appear due to physically imposed folding and tearing on the curved surface. It can also appear due to concentrations in growth or thermal strains on C [11]. The plastic strain fields will be shown to appear as sources of incompatibility for the elastic fields and thereby for the generation of internal stress and moment fields.

The distributional elastic stretching and bending strains, $\boldsymbol{E}^e \in \mathcal{B}(S, \text{Sym})$ and $\boldsymbol{\Lambda}^e \in \mathcal{B}(S, \text{Sym})$, are defined as

$$\boldsymbol{E}^e(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{e}, \boldsymbol{\psi}\rangle \,\text{da} \text{ and } \boldsymbol{\Lambda}^e(\boldsymbol{\psi}) = \int_S \langle \boldsymbol{\lambda}, \boldsymbol{\psi}\rangle \,\text{da}, \tag{59}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(S, \text{Lin})$, where $\boldsymbol{e}$ and $\boldsymbol{\lambda}$ are piecewise smooth (possibly discontinuous across C) and uniformly bounded symmetric tensor fields on S. The distributional plastic stretching and bending strains, $\boldsymbol{E}^p \in \mathcal{C}(S, \text{Sym})$ and $\boldsymbol{\Lambda}^p \in \mathcal{D}'(S, \text{Sym})$, are defined as

$$\boldsymbol{E}^p(\boldsymbol{\psi}) = \int_C \langle \boldsymbol{e}_C, \boldsymbol{\psi}\rangle \,\text{dl} \text{ and } \boldsymbol{\Lambda}^p(\boldsymbol{\psi}) = \int_C \langle \boldsymbol{\gamma}, \boldsymbol{\psi}\rangle \,\text{dl} + \int_C \langle \boldsymbol{\gamma}_1, \partial\boldsymbol{\psi}/\partial\nu\rangle \,\text{dl}, \tag{60}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(S, \text{Lin})$, where $\boldsymbol{e}_c$, $\boldsymbol{\gamma}$, and $\boldsymbol{\gamma}_1$ are smooth and uniformly bounded symmetric tensor fields on C. The distributional total stretching and bending strains are composed of elastic and plastic components, i.e.,

$$\boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^p \text{ and } \boldsymbol{\Lambda} = \boldsymbol{\Lambda}^e + \boldsymbol{\Lambda}^p. \tag{61}$$

While the total stretching and bending strains are necessarily compatible, as demanded by reasonable kinematics, the elastic and plastic components are possible incompatible. The notion of compatibility and incompatibility (lack of compatibility) are taken to be in the sense of the discussion in the preceding section on imperfectly bonded interfaces.

The compatibility of the total strain requires (36) where $\boldsymbol{L}$ is as given in (31). We can use (61) and the linearity of $\boldsymbol{L}$ to write the decomposition

$$\boldsymbol{L} = \boldsymbol{L}^e + \boldsymbol{L}^p, \text{ where} \tag{62a}$$

$$\boldsymbol{L}^e = \boldsymbol{n} \times \boldsymbol{\Lambda}^e - (\boldsymbol{n} \times \boldsymbol{E}^e)\boldsymbol{b} - \boldsymbol{n} \otimes (\mathbb{P}\,\mathrm{Div}_S(\boldsymbol{E}^e \times \boldsymbol{n})) \text{ and} \tag{62b}$$

$$\boldsymbol{L}^p = \boldsymbol{n} \times \boldsymbol{\Lambda}^p - (\boldsymbol{n} \times \boldsymbol{E}^p)\boldsymbol{b} - \boldsymbol{n} \otimes (\mathbb{P}\,\mathrm{Div}_S(\boldsymbol{E}^p \times \boldsymbol{n}))\,. \tag{62c}$$

The compatibility condition (36) can therefore be written as

$$\mathrm{Div}_S(\boldsymbol{L}^e \times \boldsymbol{n}) = \boldsymbol{N}, \text{ where} \tag{63a}$$

$$\boldsymbol{N} = \mathrm{Div}_S(\boldsymbol{L}^p \times \boldsymbol{n}) \tag{63b}$$

is the distributional incompatibility field supported on C. We can evaluate its form using (62c) as

$$\boldsymbol{N}(\boldsymbol{\psi}) = \int_C \langle \boldsymbol{\eta}_1, \boldsymbol{\psi} \rangle + \langle \boldsymbol{\eta}_2, \nabla\boldsymbol{\psi}\boldsymbol{\nu} \rangle + \langle \boldsymbol{\eta}_3, \nabla\nabla\boldsymbol{\psi}(\boldsymbol{\nu}\otimes\boldsymbol{\nu}) \rangle \,\mathrm{dl}, \text{ where} \tag{64a}$$

$$\begin{aligned}\boldsymbol{\eta}_1 = {}& \frac{dk_g}{ds}(\boldsymbol{n} \times \gamma_1\boldsymbol{\nu}) - \frac{dk_g}{ds}\langle \boldsymbol{e}_C, \boldsymbol{t}\otimes\boldsymbol{\nu}\rangle \boldsymbol{n} + k_g\boldsymbol{n} \times \left(\frac{d\boldsymbol{\gamma}_1}{ds}\boldsymbol{\nu}\right) - k_g\frac{d\boldsymbol{n}}{ds}\langle \boldsymbol{e}_C, \boldsymbol{t}\otimes\boldsymbol{\nu}\rangle \\ & + k_g\frac{d\boldsymbol{n}}{ds} \times \gamma_1\boldsymbol{\nu} - k_g\boldsymbol{n}\left\langle \frac{d(\boldsymbol{e}_C\boldsymbol{t})}{ds}, \boldsymbol{\nu}\right\rangle + \frac{d}{ds}(\boldsymbol{n} \times \boldsymbol{\gamma}\boldsymbol{\nu}) + \\ & + \frac{d}{ds}\left(\frac{\partial\boldsymbol{n}}{\partial\nu} \times \gamma_1\boldsymbol{\nu} - \frac{\partial\boldsymbol{n}}{\partial\nu}\langle \boldsymbol{e}_C, \boldsymbol{t}\otimes\boldsymbol{\nu}\rangle - \boldsymbol{n} \times \boldsymbol{e}_C(\boldsymbol{b}\boldsymbol{\nu}) - \boldsymbol{n}\left\langle \frac{d\boldsymbol{e}_C}{ds}, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\right\rangle + k_g\boldsymbol{n}\langle \boldsymbol{e}_C, \boldsymbol{t}\otimes\boldsymbol{\nu}\rangle\right),\end{aligned} \tag{64b}$$

$$\begin{aligned}\boldsymbol{\eta}_2 = {}& -\boldsymbol{n}\left(2\left\langle \frac{d\boldsymbol{e}_C}{ds}, \boldsymbol{\nu}\otimes\boldsymbol{t}\right\rangle + k_g\langle \boldsymbol{e}_C, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\rangle + k_n\langle \gamma_1, \boldsymbol{\nu}\otimes\boldsymbol{\nu}\rangle\right) + \boldsymbol{n} \times \boldsymbol{\gamma}\boldsymbol{t} - (\boldsymbol{n} \times \boldsymbol{e}_C)(\boldsymbol{b}\boldsymbol{t}) \\ & + \boldsymbol{n} \times \left(\frac{d\boldsymbol{\gamma}_1}{ds}\boldsymbol{\nu}\right) - \frac{d\boldsymbol{n}}{ds}\langle \boldsymbol{e}_C, \boldsymbol{t}\otimes\boldsymbol{\nu}\rangle\,, \text{ and}\end{aligned} \tag{64c}$$

$$\boldsymbol{\eta}_3 = \boldsymbol{n} \times \gamma_1\boldsymbol{t} - \boldsymbol{n}\langle \boldsymbol{e}_C, \boldsymbol{t}\otimes\boldsymbol{t}\rangle\,, \tag{64d}$$

for $\boldsymbol{\psi} \in \mathcal{D}(S, \mathbb{R}^3)$. On using (62b) and (64a) in (63a) we can obtain the local incompatibility equations for bulk elastic strains, all on C, as

$$\boldsymbol{n} \times ([\![\boldsymbol{\lambda}]\!]\boldsymbol{t} - [\![\boldsymbol{e}]\!]\boldsymbol{b}\boldsymbol{t}) - \boldsymbol{n}(\langle [\![\mathrm{div}_S(\boldsymbol{e} \times \boldsymbol{n})]\!], \boldsymbol{t}\rangle) - \frac{d}{ds}\boldsymbol{n}\langle [\![\boldsymbol{e}]\!], \boldsymbol{t}\otimes\boldsymbol{\nu}\rangle = -\boldsymbol{\eta}_1, \tag{65a}$$

$$-\boldsymbol{n}(\langle [\![\boldsymbol{e}]\!], \boldsymbol{t}\otimes\boldsymbol{t}\rangle) = -\boldsymbol{\eta}_2, \text{ and} \tag{65b}$$

$$\boldsymbol{\eta}_3 = \boldsymbol{0}. \tag{65c}$$

The condition (65c) imposes restrictions on the interfacial plastic concentrations. These equations when combined with (47a) and written in terms of bulk stresses and moments provide the governing equations, in addition to the stress equilibrium equations, for the determination of the bulk fields. We frame one such boundary value problem next.

A boundary value problem. We consider a simply connected plate S with an interfacial curve C such that $\partial C - \partial S = \emptyset$. Let $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ to be piecewise smooth (possibly discontinuous across C) uniformly bounded symmetric tensor fields on S, representing the bulk stress and the bulk moment field on S, respectively. In the absence of external forces, they satisfy the following equilibrium equations [10]

$$\operatorname{div}_S \boldsymbol{\sigma} = \mathbf{0} \text{ and } \operatorname{div}_S \operatorname{div}_S \boldsymbol{m} = 0 \text{ in } S - C, \tag{66a}$$

$$[\![\boldsymbol{\sigma}\boldsymbol{\nu}]\!] = 0, \ \langle [\![\boldsymbol{m}]\!], \boldsymbol{\nu} \otimes \boldsymbol{\nu} \rangle = 0, \text{ and } \langle [\![\operatorname{div}_S \boldsymbol{m}]\!], \boldsymbol{\nu} \rangle = 0 \text{ on } C, \tag{66b}$$

along with boundary conditions

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{0}, \ \langle \boldsymbol{m}, \boldsymbol{\nu} \otimes \boldsymbol{\nu} \rangle = 0, \text{ and } \langle \operatorname{div}_S \boldsymbol{m}, \boldsymbol{\nu} \rangle = 0 \ \text{ on } \partial S. \tag{67}$$

The stress and bending moment are constitutively associated with the bulk elastic strains through

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{e} \text{ and } \boldsymbol{m} = \mathbb{D}\boldsymbol{\lambda}, \tag{68}$$

where $\mathbb{C}$ and $\mathbb{D}$ represent the stretching and bending moduli tensors, respectively. The bulk elastic strains satisfy (47a) and (65) which, when written in terms of stress and moment fields, yield

$$\operatorname{div}_S(\operatorname{div}_S(\mathbb{C}^{-1}\boldsymbol{\sigma} \times \boldsymbol{n}) \times \boldsymbol{n}) = 0 \text{ in } S - C, \tag{69a}$$

$$\operatorname{div}_S(\mathbb{D}^{-1}\boldsymbol{m} \times \boldsymbol{n}) = \mathbf{0} \text{ in } S - C, \tag{69b}$$

$$\boldsymbol{n} \times ([\![\mathbb{D}^{-1}\boldsymbol{m}]\!]\boldsymbol{t}) - \boldsymbol{n}(\langle [\![\operatorname{div}_S(\mathbb{C}^{-1}\boldsymbol{\sigma} \times \boldsymbol{n})]\!], \boldsymbol{t} \rangle) - \frac{d}{ds}\boldsymbol{n} \left\langle [\![\mathbb{C}^{-1}\boldsymbol{\sigma}]\!], \boldsymbol{t} \otimes \boldsymbol{\nu} \right\rangle = -\boldsymbol{\eta}_1 \text{ on } C, \text{ and} \tag{69c}$$

$$\boldsymbol{n}(\left\langle [\![\mathbb{C}^{-1}\boldsymbol{\sigma}]\!], \boldsymbol{t} \otimes \boldsymbol{t} \right\rangle) = \boldsymbol{\eta}_2 \text{ on } C. \tag{69d}$$

Equations (66), (69), and (67) constitute a complete boundary value problem for the determination of $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ for a given prescription of interfacial plastic strain concentrations.

A de Rham's theorem

For a smooth orientable surface $S \subset \mathbb{R}^3$, let ω be a smooth p form on S; $d\omega$ then is a smooth $(p+1)$ form representing the exterior derivative of ω [2]. The exterior product of a p form ω and a q form $\bar{\omega}$ is given by a $(p+q)$ form $\omega \wedge \bar{\omega}$. Let (x^1, x^2) be a coordinate system for any local patch of the surface with immersion in $\mathbb{R}^3$ given by $\boldsymbol{\chi}(x^1, x^2)$. The natural basis of the tangent space is given by $\boldsymbol{a}_\alpha = \partial \boldsymbol{\chi} / \partial x^\alpha$; the dual basis $\boldsymbol{a}^\alpha$ is defined such that $\langle \boldsymbol{a}_\alpha, \boldsymbol{a}^\beta \rangle = \delta^\alpha_\beta$.

A smooth 0 form on S uniquely defines a scalar field on the surface. A smooth 1 form ω can be represented as $\omega = \omega_1 dx_1 + \omega_2 dx_2$. Any smooth 1 form ω can be associated uniquely with a smooth tangential vector field $\boldsymbol{v}_\omega$, with $\langle \boldsymbol{v}_\omega, \boldsymbol{n} \rangle = 0$, such that $\boldsymbol{v}_\omega = \omega_1 \boldsymbol{a}^1 + \omega_2 \boldsymbol{a}^2$. A smooth 2 form $\omega = \omega_0 dx^1 \wedge dx^2$ can be uniquely associated with the scalar field given by $\phi_\omega = \omega_0 \langle \boldsymbol{a}^1 \times \boldsymbol{a}^2, \boldsymbol{n} \rangle$. The representation of smooth forms in terms of scalar and tangential vector fields is coordinate invariant and extendable to the entire surface. For the rest of this Appendix we will represent the smooth form and the corresponding scalar or tangential vector field by the same symbol. For a smooth 0 form ϕ and a smooth 1 form $\boldsymbol{\omega}$ we have

$$d\phi = \nabla_S \phi \text{ and } d\boldsymbol{\omega} = \operatorname{div}_S(\boldsymbol{\omega} \times \boldsymbol{n}). \tag{70}$$

The space of compactly supported smooth p forms on S is given by $\mathcal{D}_p(S)$. A p current on S is defined as a linear continuous function on $\mathcal{D}_{2-p}(S)$. The space of p currents is given by $\mathcal{D}'_p(S)$. The exterior derivative of a p current T is a $(p+1)$ current dT such that

$$dT(u) = (-1)^{p+1}T(du), \tag{71}$$

for all $u \in \mathcal{D}_{1-p}(S)$.

A 0 current Φ can be associated with a scalar distribution T_Φ such that $T_\Phi(\psi) = T(\psi\,\mathrm{da})$, where da is the smooth area 2 form. A 1 current Φ can be uniquely associated with a tangential distribution $\boldsymbol{T}_\Phi$ such that $\boldsymbol{T}_\Phi(\boldsymbol{\psi} \times \boldsymbol{n}) = \Phi(\psi_1 dx^1 + \psi_2 dx^2)$, where $\mathbb{P}\boldsymbol{\psi} = \psi_1 \boldsymbol{a}^1 + \psi_2 \boldsymbol{a}^2$. A 2 current Φ can be uniquely associated with a scalar distribution T_Φ such that $T_\Phi(\psi) = T(\psi)$. For a 0 current Φ and a 1 current $\boldsymbol{\Omega}$ we have

$$d\Phi = \nabla_S \Phi \text{ and } d\boldsymbol{\Omega} = \mathrm{div}_S(\boldsymbol{\Omega} \times \boldsymbol{n}). \tag{72}$$

A current $T \in \mathcal{D}'_p(S)$ is said to be closed if $dT = 0$; T is an exact current if there exists a $(p-1)$ current $\omega \in \mathcal{D}'_{p-1}(S)$ such that $T = d\omega$. Two p currents T_1 and T_2 are said to be homologous to each other if there exists a current ω such that $T_1 - T_2 = d\omega$. We can now state the following theorem of de Rham [12]

Theorem A.1. *A current T is homologous to zero if and only if $T(u) = 0$ for all closed smooth forms u with compact support.*

Given a tangential vector field $\boldsymbol{T} \in \mathcal{D}'(S, \mathbb{R}^3)$, satisfying $\langle \boldsymbol{T}, \boldsymbol{n} \rangle = 0$, there exists an associated 1 current T (in the sense described above). Moreover, the 1 current T being homologous to 0, i.e., there exists a 0 current U satisfying $dU = T$, is equivalent to the existence of a scalar field $U \in \mathcal{D}'(S)$ such that $\nabla_S U = \boldsymbol{T}$. Using this correspondence, the above theorem now implies Lemma 2.3.

B Constancy result

For distributions in a connected open set of a Euclidean space, the gradient of the distribution being equal to zero is equivalent to the distribution being a constant distribution [6]. For instance, with $\bar{S} \subset \mathbb{R}^2$ as a connected open set and $T \in \mathcal{D}(\bar{S})$ such that $\nabla_{\bar{S}} T = \boldsymbol{0}$, we have $T(\psi) = \int_{\bar{S}} c\psi \,\mathrm{da}$ for some constant c, for any $\psi \in \mathcal{D}(S)$. We extend this result to distributions on surfaces.

Lemma B.1. *Consider a connected and smooth surface S. Given a distribution $T \in \mathcal{D}'(S)$, such that $\nabla_S T = \boldsymbol{0}$, there exists a constant $c \in \mathbb{R}$ such that $T(\phi) = \int_S c\phi \,\mathrm{da}$ for any $\phi \in \mathcal{D}(S)$.*

Proof. Let S be a smooth surface topologically equivalent to a disc and $\bar{S} \subset \mathbb{R}^2$ be a flat topological disc such that $\boldsymbol{\chi} : \bar{S} \to S$ is a smooth invertible map. Let $\{x^1, x^2\}$ be the Cartesian coordinates in $\bar{S}$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ the corresponding orthonormal frame. Let $\boldsymbol{a}_\alpha = \partial\boldsymbol{\chi}/\partial x^\alpha$. The contravariant basis $\boldsymbol{a}^\alpha$ can be defined using $\langle \boldsymbol{a}^\alpha, \boldsymbol{a}_\beta \rangle = \delta^\alpha_\beta$. Given a tangential vector field in $\bar{S}$, $\bar{\boldsymbol{\psi}} = \psi^1 \boldsymbol{e}_1 + \psi^2 \boldsymbol{e}_2$, we associate a tangential vector field in S, $\boldsymbol{\psi}(\boldsymbol{x}) = \psi^1(\boldsymbol{\chi}^{-1}(\boldsymbol{x}))\boldsymbol{a}_1 + \psi^2(\boldsymbol{\chi}^{-1}(\boldsymbol{x}))\boldsymbol{a}_2$. Let $J = \langle \boldsymbol{a}_1 \times \boldsymbol{a}_2, \boldsymbol{n} \rangle$. Note that

$$\mathrm{div}_{\bar{S}}(\bar{\boldsymbol{\psi}} J) = (\mathrm{div}_S\, \boldsymbol{\psi}) J. \tag{73}$$

Given $T \in \mathcal{D}'(S)$ we define $\bar{T} \in \mathcal{D}'(\bar{S})$ such that $T(\psi) = \bar{T}(\bar{\psi}J)$, for any $\psi \in \mathcal{D}(S)$, $\bar{\psi} \in \mathcal{D}(\bar{S})$ with $\bar{\psi} = \psi(\boldsymbol{\chi})$. We have

$$\nabla_{\bar{S}}\bar{T}(\bar{\boldsymbol{\psi}}J) = -\bar{T}((\text{div}_S\,\boldsymbol{\psi})J) = -T(\text{div}_S\,\boldsymbol{\psi}) = 0. \tag{74}$$

This implies that $\nabla_{\bar{S}}\bar{T} = \mathbf{0}$. Using the result for distributions on Euclidean spaces, we can argue existence of a constant c such that $T(\psi) = \bar{T}(\bar{\psi}J) = \int_{\bar{S}} c\bar{\psi}Jdx^1dx^2 = \int_S c\psi\,\text{da}$. This establishes our result for surface with disc topology. Given a connected and smooth surface S (of arbitrary topology), the distribution will be constant in any arbitrary local neighbourhood (topological disc) implying that the distribution will be constant over the full surface. □

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