

Multiphysics Bianisotropy in Wave–Diffusion Metamaterials

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Many physical systems, including thermoelastic solids, ionic conductors, and piezoelectric semiconductors, are governed by coupled wave-diffusion dynamics in which propagating fields coexist with dissipative transport. We develop a first-principles homogenization theory for such heterogeneous media and complement it with a scattering analysis of thermoelastic multilayers. We show that breaking spatial symmetries at finer scales induces macroscopic cross-couplings between otherwise uncoupled diffusive and propagating fields. Their signatures include strongly direction-dependent thermomechanical cross-channel scattering, offering pathways for signal manipulation in wave–diffusion metamaterials.

Symmetry is one of the most fundamental organizing principles in physics, constraining which physical fields can couple and which macroscopic material responses are allowed. In metamaterials [1–4], these constraints can be selectively relaxed by breaking spatial symmetries at subwavelength scales, thereby activating effective constitutive couplings absent in the underlying constituents [5]. Prominent examples in wave physics include electromagnetic bianisotropy [6–8], Willis coupling in elastodynamics [9–11], and electromomentum coupling in piezoelectric composites [12–15]. Analogous bianisotropic effects have recently been identified in diffusion physics, first through spatiotemporal modulation [16, 17] and subsequently through spatial asymmetry alone [18, 19].

Many physical systems, including thermoelastic solids [20], ionic conductors [21–23], and piezoelectric semiconductors [24, 25], exhibit coupled wave-diffusion dynamics, in which propagating fields interact with dissipative transport. Whether microscopic asymmetries generate constitutive cross-couplings between disparate propagating and dissipative fields has remained largely unexplored. Here we show that the answer is yes: asymmetry introduces macroscopic wave-diffusion conversion that is absent from the constituents response, and produces direction-sensitive signatures in observable responses. These phenomena are captured by constitutive multiphysics bianisotropic terms.

Our approach is rooted in the exact homogenization theory introduced by Willis for wave physics [26–29] and later extended to coupled and diffusive systems [12, 19, 30, 31]. Here, we develop a unified framework for wave–diffusion multiphysics, from which earlier results in elastodynamics and heat conduction appear as limiting cases. The theory yields exact effective constitutive relations that are generally nonlocal in space and time. It also reveals emergent cross-couplings between propagating and dissipative fields.

To expose the observable consequences of these couplings in a simple platform, we complement the exact theory with a two-channel scattering retrieval for thermoelastic multilayers. We show that emergent bianisotropic terms govern strongly direction-dependent thermomechanical cross-

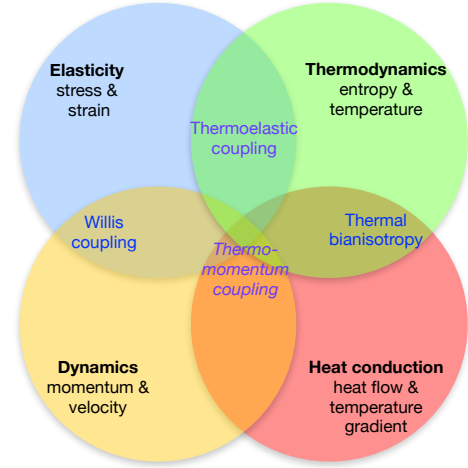


FIG. 1. Wave–diffusion multiphysics bianisotropy, illustrated using thermoelasticity: elastic fields propagate, thermal fields diffuse, and thermal expansion couples them. Microscopic symmetry breaking activates emergent macroscopic cross-couplings.

channel scattering, offering pathways for signal manipulation in wave–diffusion metamaterials.

The thermoelastic setting is summarized in Fig. 1. The diffusive part of the system is governed by the entropy equation $\nabla \cdot \mathbf{q} + \theta_R \dot{\eta} = r$, where η and r are the entropy increment and heat source per unit volume, respectively, $\mathbf{q}$ is the heat flux per unit area, and θ_R is the reference temperature. The propagative part is governed by the momentum equation $\nabla \cdot \boldsymbol{\sigma} - \dot{\mathbf{p}} = -\mathbf{f}$, where $\boldsymbol{\sigma}$ is the stress, $\mathbf{p}$ is the momentum density and $\mathbf{f}$ is the body-force density. Let

$$\mathbf{b} = (\nabla \mathbf{u}, \dot{\mathbf{u}}, \nabla \theta, \dot{\theta})^T, \quad \mathbf{h} = (\boldsymbol{\sigma}, \mathbf{p}, -\mathbf{q}, \theta_R \dot{\eta})^T, \quad (1)$$

where $\mathbf{u}$ is the displacement field, and θ is the temperature increment relative to θ_R , then the solid’s response obeys [20]

$$\mathbf{h} = \mathbf{L}(\mathbf{x})\mathbf{b}, \quad \mathbf{L} = \begin{pmatrix} \mathbf{C} & 0 & 0 & -\boldsymbol{\beta} \\ 0 & \rho \mathbf{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{\kappa} & 0 \\ \theta_R \boldsymbol{\beta}^T & 0 & 0 & c \end{pmatrix}, \quad (2)$$

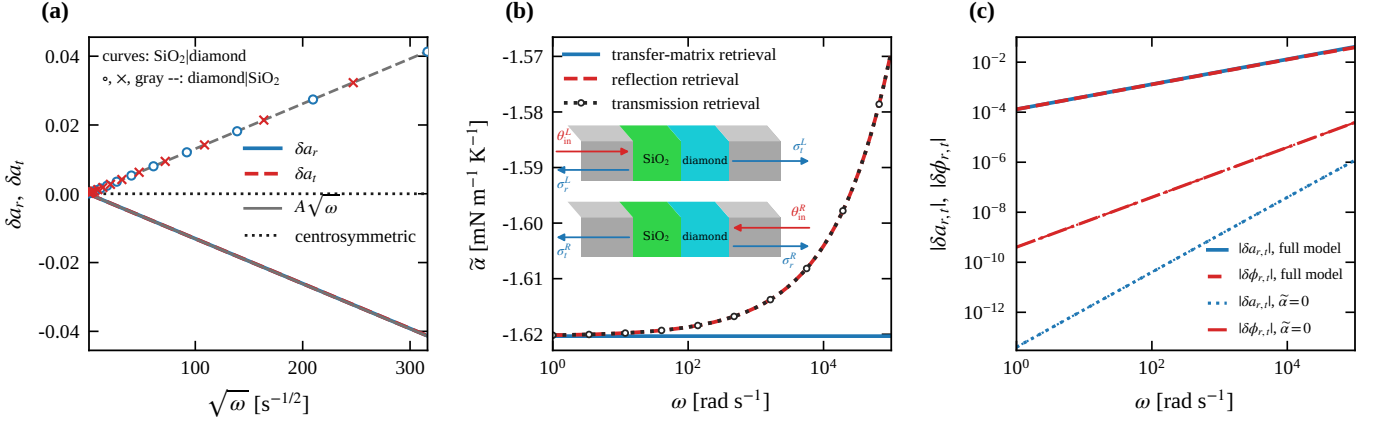


FIG. 2. Scattering fingerprint of $\tilde{\alpha}$ for a SiO₂–diamond bilayer. (a) Reflection and transmission log-magnitude contrasts, δa_r (blue solid) and δa_t (red dashed), versus $\sqrt{\omega}$. Curves denote SiO₂|diamond, open circles and crosses its reversal, the gray dashed line the leading $A\sqrt{\omega}$ law, and the black dotted line the centrosymmetric SiO₂|diamond|SiO₂ control. (b) The emergent bianisotropic term $\tilde{\alpha}$ obtained from the full transfer matrix (blue solid) and its estimates obtained by inverting the subwavelength reflection- and transmission-ratio relation in Eq. (8); the inset defines the two thermal-incidence experiments. (c) Absolute amplitude (blue) and phase (red) contrasts for reflection and transmission. Solid/dashed curves use the full retrieved model, whereas dotted/dash-dotted curves result from removing from the model the reciprocal $\tilde{\alpha}$ pair while retaining all other retrieved coefficients.

where $\mathbf{C}$, ρ , κ , β and c are the elasticity, mass density, conductivity, volumetric heat capacity and thermoelastic coupling; these parameters are functions of position $\mathbf{x}$ in a heterogeneous solid. Our objective is to expose collective response mechanisms that are obscured in the heterogeneous description by replacing it with a homogeneous one governed by effective relations between macroscopic averages of rapidly varying microscopic fields [32]. While homogenization of thermoelastic composites has previously established effective thermoelastic properties and, in dynamic settings, additional corrective and memory effects [33–36], whether spatial inversion breaking creates effective cross-couplings has remained largely unexplored.

To this end, we introduce an auxiliary residual to thereby remove the nonuniqueness that otherwise arises when gradients and their underlying potentials are algebraically linked [29, 37–39]. We formally solve the heterogeneous boundary-value problem using its Green operator, ensemble averaging, and elimination the microscopic fluctuations to obtain the exact effective constitutive operator (End Matter). The central conclusion is that the macroscopic fields obey

$$\begin{pmatrix} \langle \sigma \rangle \\ \langle \mathbf{p} \rangle \\ -\langle \mathbf{q} \rangle \\ \langle \theta_R \eta \rangle \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{C}} & \tilde{\mathbf{S}} & \tilde{\alpha} & -\tilde{\beta} \\ \tilde{\mathbf{S}}^T & \tilde{\rho} & \tilde{\gamma} & \tilde{\Gamma} \\ -\theta_{RS} \tilde{\alpha}^T & -\theta_{RS} \tilde{\gamma}^T & \tilde{\kappa} & s \tilde{\xi} \\ \theta_R \tilde{\beta}^T & -\theta_R \tilde{\Gamma}^T & \tilde{\xi}^T & \tilde{c} \end{pmatrix} \begin{pmatrix} \langle \nabla \mathbf{u} \rangle \\ \langle s \mathbf{u} \rangle \\ \langle \nabla \theta \rangle \\ \langle \theta \rangle \end{pmatrix}; \quad (3)$$

here angular brackets denote ensemble average and $\tilde{\alpha}$, $\tilde{\gamma}$, $\tilde{\Gamma}$ capture emergent bianisotropic couplings, in addition to the elastic and thermal Willis couplings $\tilde{\mathbf{S}}$ and $\tilde{\xi}$, respectively. The effective relations (3) are nonlocal in space and time and inherit the self-adjoint structure of the microscopic constitutive relations. For a statistically homogeneous medium, the nonlocal

response can be represented in wavevector space by $\tilde{\mathbf{L}}(\mathbf{k}, s)$. Spatial inversion constrains the odd couplings $\tilde{\mathbf{S}}$, $\tilde{\alpha}$, $\tilde{\Gamma}$, and $\tilde{\xi}$ to vanish at $\mathbf{k} = \mathbf{0}$ in a centrosymmetric medium, although they may remain nonzero at finite $\mathbf{k}$. Thus, in the local subwavelength limit, breaking centrosymmetry unlocks material responses that are otherwise forbidden, providing route from microscopic symmetry breaking to emergent bianisotropic physics.

In one dimension, bianisotropic terms of a subwavelength element can often be retrieved from its scattering response [10, 40], although the retrieved terms may differ from those obtained by rigorous field-averaging homogenization [13, 41]. We therefore consider a thermoelastic multilayer occupying $x \in [-l/2, l/2]$, embedded in a host medium with C_0 , ρ_0 , κ_0 , c_0 and $\beta_0 = 0$. Thus, the host supports two uncoupled channels on each side: one mechanical and one thermal. At each frequency, mechanical and thermal incidence from either side determine the 2×2 reflection and transmission matrices $\mathbf{R}^{L,R}$ and $\mathbf{T}^{L,R}$, respectively. Continuity of

$$\mathbf{y} = (\sigma, u, -q, \theta)^T \quad (4)$$

relates the scattering data to the transfer matrix $\mathbf{T}$ that relates $\mathbf{y}$ at the two ends of the multilayer. Assigning the same $\mathbf{T}$ to a uniform slab defines $\tilde{\mathbf{M}}_{\text{ret}} = l^{-1} \ln \mathbf{T}$ [42]; block inversion then yields the retrieval-based form of Eq. (3) (see End Matter). For a bilayer with ordered layer generators $\mathbf{M}_1$ and $\mathbf{M}_2$, the Baker–Campbell–Hausdorff expansion gives

$$\tilde{\mathbf{M}}_{\text{ret}} = \frac{l_1}{l} \mathbf{M}_1 + \frac{l_2}{l} \mathbf{M}_2 + \frac{l_1 l_2}{2l} [\mathbf{M}_2, \mathbf{M}_1] + \cdots, \quad (5)$$

where higher-order terms are negligible when each layer is deeply subwavelength relative to both the elastic wavelength and the thermal diffusion length at the frequency of interest. The commutator changes sign under layer reversal and generates

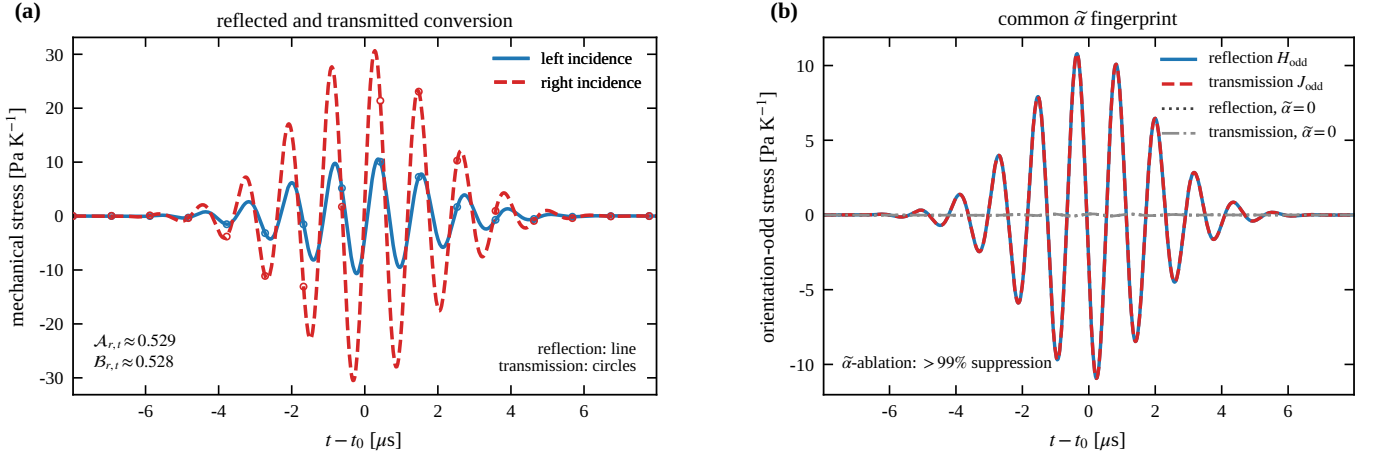


FIG. 3. Finite-bandwidth asymmetric transmission in a 96-nm SiO₂|24-nm Cu bilayer. Identical unit-peak thermal pulses are incident from opposite sides, and the abscissa is the shifted time $t - t_0$. (a) Reflected mechanical stresses are shown by lines and transmitted stresses by open circles; blue solid and red dashed curves denote left and right incidence. The directional metrics are $\mathcal{A}_{r,t} \approx 0.529$ and $\mathcal{B}_{r,t} \approx 0.528$. (b) Orientation-odd reflected and transmitted stresses associated with H_{odd} (blue solid) and J_{odd} (red dashed). Gray dotted and dash-dotted curves are the corresponding responses after removing the complete reciprocal $\tilde{\alpha}$ pair; both odd waveforms are suppressed by more than 99%.

the emergent multiphysics bianisotropic terms

$$\tilde{\Gamma}(s) = -s \frac{l_1 l_2}{2l} \frac{\check{C}}{C_1 C_2} \bar{\rho} \Delta\beta + \dots, \quad (6a)$$

$$\tilde{\alpha}(s) = \frac{l_1 l_2}{2l} \check{C} \check{\kappa} \left(\frac{\beta_1}{C_1 \kappa_2} - \frac{\beta_2}{C_2 \kappa_1} \right) + \dots, \quad (6b)$$

where $\Delta f = f_2 - f_1$, $\bar{f} = (l_1 f_1 + l_2 f_2)/l$, and $\check{f} = l/(l_1/f_1 + l_2/f_2)$. Thus, at leading order, $\tilde{\Gamma}$ is odd in frequency and $\tilde{\alpha}$ approaches a real orientation-odd constant as $s \rightarrow 0$; $\tilde{\gamma}$ first appears at the next order (End Matter).

We now expose these terms through the thermal-to-mechanical scattering of a 25-nm SiO₂|25-nm diamond bilayer. Let $r_{\times}^{L,R}$ and $t_{\times}^{L,R}$ be the mechanical reflection and transmission amplitudes generated by unit-temperature thermal incidence from the left or right. Define

$$\Lambda_r = -\frac{r_{\times}^L}{r_{\times}^R}, \quad \Lambda_t = -\frac{t_{\times}^L}{t_{\times}^R}, \quad (7)$$

$$\delta a_j = \ln |\Lambda_j|, \quad \delta \phi_j = \arg \Lambda_j, \quad j \in \{r, t\},$$

so that $\delta a_{r,t}$ is the log-amplitude asymmetry, and $\delta \phi_{r,t}$ is the phase asymmetry. In the subwavelength limit,

$$\Lambda_r \simeq \Lambda_t \simeq \frac{\tilde{\beta}\tilde{\kappa} + \tilde{\alpha}/Z_{0t}}{\tilde{\beta}\tilde{\kappa} - \tilde{\alpha}/Z_{0t}}, \quad Z_{0t} = \frac{1}{\sqrt{s\kappa_0 c_0}}, \quad (8)$$

so that, for real quasistatic coefficients and $s = i\omega$,

$$\delta a_{r,t} \simeq \delta \phi_{r,t} = A\sqrt{\omega} [1 + o(1)], \quad A = \sqrt{2} \frac{\tilde{\alpha}(0)\sqrt{\kappa_0 c_0}}{\check{C} \check{\kappa} (\beta/C)}, \quad (9)$$

Figure 2(a) shows δa_r and δa_t versus $\sqrt{\omega}$ as calculated from the exact transfer matrix for the SiO₂|diamond bilayer, its reversal, and a centrosymmetric SiO₂|diamond|SiO₂ control

of the same length. The contrasts follow the low-frequency scaling $A\sqrt{\omega}$, change sign under layer reversal, and vanish for the centrosymmetric benchmark. Panel (b) retrieves $\tilde{\alpha}$ from the full transfer matrix and compares it with estimates obtained by inverting the subwavelength reflection- and transmission-ratio relation in Eq. (8). The three calculations agree within 1% through $5 \times 10^3 \text{ rad s}^{-1}$. Panel (c) isolates the role of $\tilde{\alpha}$ by comparing $|\delta a_{r,t}|$ and $|\delta \phi_{r,t}|$ from the full effective model with those obtained after setting the entries associated with $\tilde{\alpha}$ to zero, while retaining all other terms from the retrieval procedure. The contrasts then collapse by orders of magnitude, demonstrating that the effect is not merely orientation-dependent thermoelastic conversion: its dominant directional component is encoded by the emergent coupling $\tilde{\alpha}$.

Having established the harmonic scattering fingerprint of multiphysics bianisotropy, we next examine whether it produces a significant effect in a finite thermoelastic signal. To this end, we use the exact transfer matrix to calculate the scattered stress the scattered stress generated by identical Gaussian-modulated thermal pulses,

$$\theta_{\text{in}}(t) = \exp\left[-\frac{(t-t_0)^2}{2\tau^2}\right] \cos[\omega_c(t-t_0)], \quad (10)$$

$$\omega_c = 5.23 \times 10^6 \text{ rad s}^{-1}, \quad \tau = 1.912 \mu\text{s},$$

incident on the bilayer from opposite sides. The frequency-domain reflected and transmitted stress transfer functions are

$$\hat{H}^{L,R}(\omega) = \frac{\hat{\sigma}_r^{L,R}(\omega)}{\hat{\theta}_{\text{in}}(\omega)}, \quad \hat{J}^{L,R}(\omega) = \frac{\hat{\sigma}_t^{L,R}(\omega)}{\hat{\theta}_{\text{in}}(\omega)}; \quad (11)$$

The pulse spectrum is obtained by FFT; the exact heterogeneous transfer functions are evaluated frequency by frequency over its spectral support (negative-frequency spectrum is obtained by complex-conjugate symmetry) and an inverse FFT yields the stress histories.

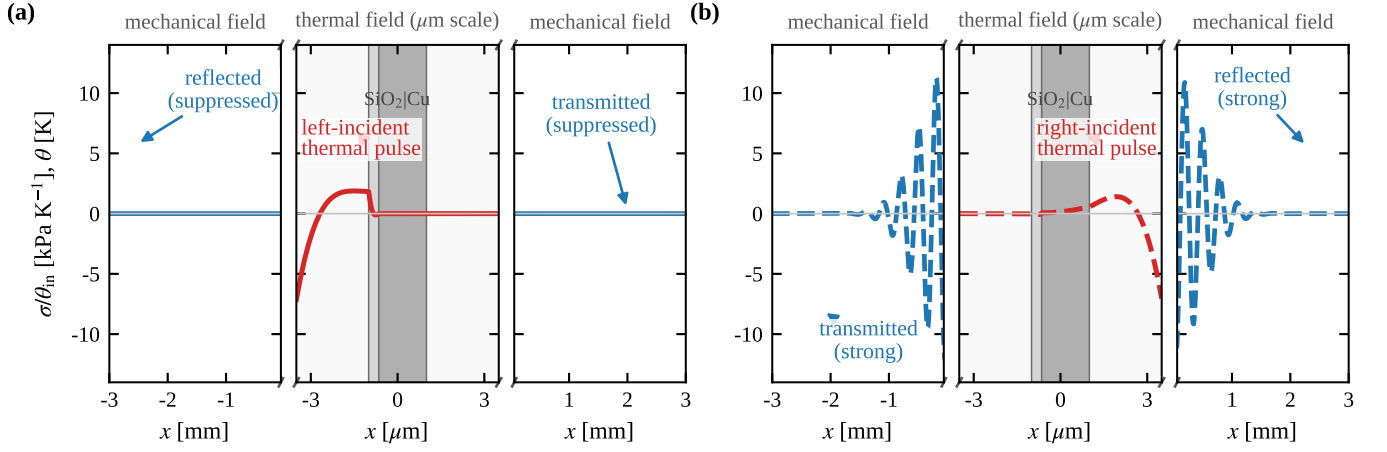


FIG. 4. Diode-like cross-conversion outside the subwavelength regime. A 350-nm SiO₂|1650-nm Cu bilayer is driven by the same pulse form as Eq. (10), but with $\omega_c = 1.544 \times 10^8 \text{ rad s}^{-1}$ and $\tau = 64.8 \text{ ns}$. Red curves show temperature and blue curves the generated mechanical stress; solid and dashed styles correspond to left and right incidence. The central micrometer-scale thermal region is magnified relative to the millimeter-scale mechanical host regions, as marked by the broken horizontal axes. (a) Left incidence lies near a thermal-to-mechanical transmission zero, suppressing reflected and transmitted stress. (b) Right incidence produces strong converted stresses. The reflected and transmitted metrics are $(\mathcal{A}_r, \mathcal{B}_r) = (0.998, 0.998)$ and $(\mathcal{A}_t, \mathcal{B}_t) = (0.999, 0.999)$, respectively.

We quantify their directional contrast by

$$\mathcal{A}_j = \frac{\max_t |\sigma_j^L - \sigma_j^R|}{\max_t |\sigma_j^L| + \max_t |\sigma_j^R|}, \quad \mathcal{B}_j = \frac{\|\sigma_j^L - \sigma_j^R\|_2}{\|\sigma_j^L\|_2 + \|\sigma_j^R\|_2}, \quad (12)$$

so that $\mathcal{A}_j$ measures the largest instantaneous directional contrast, whereas $\mathcal{B}_j$ measures the difference between the complete waveforms, $j = r, t$, both normalized between zero and unity.

Searching for a composition that maximizes these metrics while satisfying the local-regime constraints over the band containing 99.9% of the pulse energy led to the selection 96-nm SiO₂|24-nm Cu bilayer, for which $\mathcal{A}_{r,t} \approx 0.529$ and $\mathcal{B}_{r,t} \approx 0.528$. In this regime, $|k_m| \leq 0.1$ and $|k_{th}| \leq 0.35$, where k_m and k_{th} are the largest elastic and thermal wavenumbers in the bilayer. Thus the calculation remains within the local subwavelength regime, although not within the stricter $\omega \rightarrow 0$ range. Figure 3(a) depicts the reflected and transmitted mechanical stresses generated by the incident pulses as functions of the shifted time $t - t_0$. For a fixed incidence direction, the reflected and transmitted responses nearly coincide, whereas reversing the thermal incidence produces a pronounced change in both amplitude and temporal phase. Thus, the same directional cross-channel asymmetry appears in reflection and transmission, demonstrating strongly asymmetric yet reciprocal thermal-to-mechanical transmission. Panel (b) compares the orientation-odd histories generated by $H_{\text{odd}} = (H^L - H^R)/2$ and $J_{\text{odd}} = (J^L - J^R)/2$. Their close agreement reflects the common leading $\tilde{\alpha}$ fingerprint of cross reflection and cross transmission. Removing the reciprocal $\tilde{\alpha}$ pair from the constitutive model suppresses both H_{odd} and J_{odd} by more than 99%, linking the subwavelength stress asymmetry directly to that emergent term.

The pronounced directional contrast in Fig. 3 naturally raises a broader question: can asymmetric thermomechanical conver-

sion be pushed to the extreme of a 'diode-like' response? By 'diode-like' we describe strongly asymmetric, yet reciprocal, cross-channel transmission [43], rather than nonreciprocal isolation [44–46]. We obtain this by relaxing the subwavelength restriction and considering the exact scattering response of a thicker asymmetric bilayer. By tuning these finite-slab effects, one incidence direction can be brought close to a thermomechanical transmission zero, while the opposite direction retains strong conversion, yielding a large directional stress contrast without breaking reciprocity.

Specifically, Fig. 4 uses a 350-nm SiO₂|1650-nm Cu bilayer at $\omega_c = 1.544 \times 10^8 \text{ rad s}^{-1}$ (outside the local homogenization regime), showing a time-snapshot of the thermoelastic response (see also Supplementary Video S1). Panels (a) and (b) correspond to left and right incidence, respectively. The central region in each panel is a magnified view of the thermal field, θ (red), over a micrometer-scale coordinate containing the 2 μm laminate, whereas the outer regions show the generated mechanical stress, $\sigma/\theta_{\text{in}}$ (blue), over a millimeter-scale coordinate; the breaks in the horizontal axis therefore separate the thermal near field from the much longer elastic propagation scale. A strong directional contrast is evident: for left incidence the reflected and transmitted stresses are nearly suppressed, while right incidence produces pronounced scattered stress pulses. This is an outcome of a strongly asymmetric thermal penetration with the large thermoelastic contrast of the constituents: at ω_c , the thermal diffusion lengths $\delta_t = \sqrt{2\kappa/(\omega_c c)}$ are approximately 0.1 μm in SiO₂ and 1.22 μm in Cu, while $\beta_{\text{Cu}}/\beta_{\text{SiO}_2} \approx 54$. Accordingly, thermal incidence from the SiO₂ side strongly attenuates before reaching the Cu layer, whereas incidence from the Cu side drives a much larger elastic response. The chosen thicknesses and frequency further tune the radiated stresses to nearly cancel for one incidence direction, producing the

cross-transmission zero; reversal changes the internal thermal field and removes this cancellation.

In summary, we have shown that spatial asymmetry generates multiphysics bianisotropy in coupled wave–diffusion media: emergent effective couplings between propagating and diffusive fields that are absent from the microscopic physics. We identify this constitutive structure through a unified homogenization framework and expose its measurable directional signatures through scattering analysis of thermoelastic multilayers. These signatures include asymmetric cross-channel transmission and strongly direction-dependent pulse conversion governed by the emergent couplings. Beyond the local subwavelength regime, we further show that finite-slab interference can drive the thermomechanical transmission close to zero for one incidence direction while retaining strong conversion for the opposite direction, yielding a diode-like response without violating reciprocity. More broadly, our framework provides a route to designing metamaterials with unconventional couplings in other wave–diffusion systems, including ionic, thermoelectric, and piezoelectric-semiconductor media.

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Data availability. The data that support the findings of this article are openly available [47]; embargo periods may apply.

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END MATTER

Appendix A: Source-driven homogenization.— After Laplace transformation in time, with $s = i\omega$ for harmonic fields, we introduce the potentials $\mathbf{w}$, sources $\mathbf{g}$, kinetic fields $\mathbf{h}$, and the differential operators $\mathbf{B}$ and $\mathbf{D}^\top$ as

$$\mathbf{w} = \begin{pmatrix} \mathbf{u} \\ \theta \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \mathbf{f} \\ r \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} \sigma \\ \mathbf{p} \\ -\mathbf{q} \\ \theta_R \eta \end{pmatrix}, \quad (\text{A1})$$

$$\mathbf{D}^\top = \begin{pmatrix} \nabla \cdot & -s & 0 & 0 \\ 0 & 0 & \nabla \cdot & -s \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \nabla & 0 \\ s & 0 \\ 0 & \nabla \\ 0 & 1 \end{pmatrix}.$$

Thus, $\mathbf{B}\mathbf{w}$ is the kinematic state defined in Eq. (1), while $\mathbf{h}$ is its kinetic counterpart. Using the microscopic constitutive array $\mathbf{L}(\mathbf{x})$ of Eq. (2), the heterogeneous boundary-value problem is written compactly as

$$\mathbf{h} = \mathbf{L}\mathbf{B}\mathbf{w}, \quad \mathbf{D}^\top \mathbf{L}\mathbf{B}\mathbf{w} + \mathbf{g} = 0. \quad (\text{A2})$$

We allow the microstructure to be random. A realization is labeled by $\zeta \in \mathcal{S}$, where $\mathcal{S}$ is a sample space endowed with a probability measure $p(d\zeta)$; thus, for example, the ensemble averaging of $\mathbf{L}(\mathbf{x}, \zeta)$ is defined by

$$\langle \mathbf{L} \rangle(\mathbf{x}) = \int_{\mathcal{S}} \mathbf{L}(\mathbf{x}, \zeta) p(d\zeta). \quad (\text{A3})$$

The applied sources and the auxiliary field introduced below are taken to be deterministic with respect to the ensemble.

We represent the solution of Eq. (A2) by the Green function of each heterogeneous realization. To this end, we introduce the adjoint of the operator $\mathbf{D}^\top \mathbf{L}\mathbf{B}$ with respect to the bilinear pairing $(\mathbf{a}, \mathbf{b}) = \int_{\Omega} \mathbf{a}^\top \mathbf{b} d\mathbf{x}$. Integration by parts gives a Green identity whose boundary terms contain the conjugate mechanical and thermal variables. Choosing mutually adjoint homogeneous boundary conditions then gives the reciprocity relation between the Green function $\mathbf{G}$ and its adjoint, $\mathbf{G}^\dagger(\mathbf{x}, \mathbf{x}') = \mathbf{G}^\top(\mathbf{x}', \mathbf{x})$. Applying the same Green identity to the physical fields and $\mathbf{G}^\dagger$ yields an exact representation of $\mathbf{w}$ in terms of the prescribed body and boundary sources.

The effective constitutive relations are not uniquely determined if the effective kinematic variables are constrained to remain derivatives of the same averaged potentials. We therefore introduce an independently prescribed residual kinematic field $\mathbf{m}$ so that

$$\mathbf{h} = \mathbf{L}(\mathbf{B}\mathbf{w} - \mathbf{m}), \quad \mathbf{D}^\top \mathbf{L}(\mathbf{B}\mathbf{w} - \mathbf{m}) + \mathbf{g} = 0. \quad (\text{A4})$$

The residual $\mathbf{m}$ is sure with respect to the ensemble and makes the macroscopic kinematic components independently excitable, thereby selecting a unique representative of the nonlocal constitutive class. Substitution of Eq. (A4) into the Green representation described above gives

$$\mathbf{w} = \langle \mathbf{w} \rangle + \mathbf{G} \left\{ \mathbf{g}, \begin{pmatrix} \sigma \cdot \mathbf{n} \\ -\mathbf{q} \cdot \mathbf{n} \end{pmatrix} \right\} + (\mathbf{D}^\top \mathbf{G}^\dagger)^\dagger \mathbf{L}(\mathbf{B}\langle \mathbf{w} \rangle - \mathbf{m}), \quad (\text{A5})$$

where $\mathbf{G} = \mathbf{G}(\zeta)$ is the integral operator associated with the Green function of each realization, and $(\mathbf{D}^\top \mathbf{G}^\dagger)^\dagger$ denotes the corresponding integral operator obtained after integration by parts.

Equation (A5) still contains the amplitudes of the prescribed body and boundary sources. Ensemble averaging eliminates them, yielding an expression for $\left\{ \mathbf{g}, \begin{pmatrix} \mathbf{n} \cdot \sigma \\ -\mathbf{n} \cdot \mathbf{q} \end{pmatrix} \right\}$, which upon substitution back into Eq. (A5) provides

$$\mathbf{w} = \langle \mathbf{w} \rangle - \mathbf{G}\langle \mathbf{G} \rangle^{-1} \left\{ (\mathbf{D}^\top \mathbf{G}^\dagger)^\dagger \mathbf{L} \right\} (\mathbf{B}\langle \mathbf{w} \rangle - \mathbf{m}) + (\mathbf{D}^\top \mathbf{G}^\dagger)^\dagger \mathbf{L}(\mathbf{B}\langle \mathbf{w} \rangle - \mathbf{m}). \quad (\text{A6})$$

Finally, inserting Eq. (A6) Eq. (A4) and ensemble averaging gives

$$\langle \mathbf{h} \rangle = \langle \mathbf{L}(\mathbf{B}\mathbf{w} - \mathbf{m}) \rangle = \tilde{\mathbf{L}}(\mathbf{B}\langle \mathbf{w} \rangle - \mathbf{m}). \quad (\text{A7})$$

Equation (A7) is the exact source-driven effective constitutive relation. It relates averages of the *actual* microscopic kinetic fields to averages of the *actual* microscopic kinematic fields; the resulting operator is generally nonlocal in space and time. This is distinct from assigning a homogeneous constitutive array to a finite slab solely because it reproduces the same boundary scattering.

For a periodic medium, the ensemble may be generated by translating the unit cell through all possible origins. In one dimension with period l , the sample parameter can be taken as a shift $\zeta \in [0, l)$ with uniform measure, so that

$$\langle \mathbf{F} \rangle = \frac{1}{l} \int_0^l \mathbf{F}(\zeta) d\zeta. \quad (\text{A8})$$

This translation ensemble removes any dependence on the arbitrary choice of unit-cell origin and makes the effective operator translation invariant. After Bloch–Fourier transformation it is therefore represented by $\tilde{\mathbf{L}}(k, s)$. For a one-dimensional periodic medium of period l , let k denote the Bloch wavenumber. One route to calculate $\tilde{\mathbf{L}}(k, s)$ is to write the potentials in Bloch form as

$$\mathbf{w}(x) = e^{ikx} \mathbf{w}_{\text{per}}(x), \quad \mathbf{w}_{\text{per}}(x) = \sum_n \mathbf{w}_n e^{iG_n x},$$

where $\mathbf{w}_{\text{per}}(x + l) = \mathbf{w}_{\text{per}}(x)$ and

$$G_n = \frac{2\pi n}{l}, \quad K_n = k + G_n, \quad L_n = \frac{1}{l} \int_0^l \mathbf{L}(x) e^{-iG_n x} dx,$$

$$\mathbf{B}_n = \begin{pmatrix} iK_n & 0 \\ s & 0 \\ 0 & iK_n \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}_n^\top = \begin{pmatrix} iK_n & -s & 0 & 0 \\ 0 & 0 & iK_n & -s \end{pmatrix}. \quad (\text{A9})$$

The $n = 0$ coefficients are the amplitudes of the effective fields, and $n \neq 0$ describe their microscopic periodic fluctuations. In particular,

$$\bar{\mathbf{w}} \equiv \mathbf{w}_0, \quad \langle \mathbf{w} \rangle(x) = \bar{\mathbf{w}} e^{ikx}.$$

Writing likewise $\bar{\mathbf{m}} \equiv \mathbf{m}_0$, the independently prescribed macroscopic kinematic amplitude is

$$\bar{\mathbf{b}} = \mathbf{B}_0 \bar{\mathbf{w}} - \bar{\mathbf{m}}.$$

Setting the nonzero source harmonics to zero gives, for $n \neq 0$,

$$\sum_{m \neq 0} \mathbf{A}_{nm} \mathbf{w}_m = -\mathbf{D}_n^\top \mathbf{L}_n \bar{\mathbf{b}}, \quad \mathbf{A}_{nm} = \mathbf{D}_n^\top \mathbf{L}_{n-m} \mathbf{B}_m. \quad (\text{A10})$$

Thus, $\mathbf{A}$ is the block operator that couples the nonzero potential harmonics through the mechanical and thermal balance equations. The amplitude of the effective kinetic field is

$$\bar{\mathbf{h}} = \mathbf{L}_0 \bar{\mathbf{b}} + \sum_{m \neq 0} \mathbf{L}_{-m} \mathbf{B}_m \mathbf{w}_m, \quad \langle \mathbf{h} \rangle(x) = \bar{\mathbf{h}} e^{ikx}.$$

Eliminating the nonzero harmonics using Eq. (A10) therefore yields

$$\tilde{\mathbf{L}}_{\text{SD}}(k, s) = \mathbf{L}_0 - \sum_{m, n \neq 0} \mathbf{L}_{-m} \mathbf{B}_m [\mathbf{A}^{-1}]_{mn} \mathbf{D}_n^\top \mathbf{L}_n. \quad (\text{A11})$$

Equation (A11) is the exact bulk source-driven operator at the prescribed (k, s) . Its dependence on k and s represents spatial and temporal nonlocality, respectively.

Figure 5 compares the calculation of $\tilde{\alpha}_{\text{SD}}$ at $\omega = 10^4 \text{ rad s}^{-1}$, for a bulk using the source-driven method (using $N = 384$ at which results converge), with the calculation of finite-slab scattering retrieval method, $\tilde{\alpha}_{\text{RT}}$, for three cyclic representations of the same periodic $A|B|C$ medium and their mirror images, where $A = \text{SiO}_2$, $B = \text{diamond}$, and $C = \text{Cu}$, with equal layer thicknesses $d_A = d_B = d_C =$

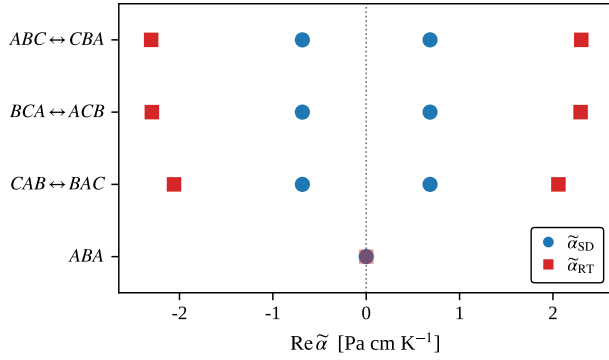


FIG. 5. Bulk source-driven computation of $\tilde{\alpha}_{SD}$ (circles) and finite-slab scattering-retrieved computation $\tilde{\alpha}_{RT}$ (squares) for cyclic representations of the periodic $A|B|C$ laminate and their mirror images. The source-driven values have identical magnitude for ABC , BCA , and CAB , demonstrating invariance to the choice of periodic unit-cell origin, whereas the retrieved values depend on the finite-slab termination. The symmetric ABA control yields $\tilde{\alpha} = 0$ in both descriptions.

20 nm. Each mirror pair is shown on the same row, making the inversion-odd character of both coefficients explicit: spatial reversal changes the sign of $\tilde{\alpha}$. The source-driven values, however, have the same magnitude for ABC , BCA , and CAB , showing that $\tilde{\alpha}_{SD}$ is invariant under a translation of the periodic unit-cell origin. In contrast, the retrieved values vary among these three finite slabs, demonstrating their sensitivity to termination. The symmetric ABA control lies at $\text{Re } \tilde{\alpha} = 0$ for both descriptions. The comparison thus distinguishes a translation-invariant bulk constitutive coupling from a scattering-equivalent finite-slab parameter, while confirming the same inversion-odd symmetry in both constructions.

Appendix B: Finite-slab scattering and retrieval.— Using the state y defined in Eq. (4), a standard homogeneous layer r obeys [48]

$$y_{,x} = M_r y, \quad T = T_N \cdots T_1, \quad T_r = e^{M_r l_r},$$

$$M_r = \begin{pmatrix} 0 & \rho_r s^2 & 0 & 0 \\ C_r^{-1} & 0 & 0 & \beta_r / C_r \\ s\theta_R \beta_r / C_r & 0 & 0 & s(c_r + \theta_R \beta_r^2 / C_r) \\ 0 & 0 & \kappa_r^{-1} & 0 \end{pmatrix}. \quad (\text{B1})$$

At either host reference plane write $y = Q_0 a$ for $a = (a_m^+, a_t^+, a_m^-, a_t^-)^T$, where

$$Q_0 = \begin{pmatrix} -sZ_{0m} & 0 & sZ_{0m} & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -Z_{0t}^{-1} & 0 & Z_{0t}^{-1} \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad (\text{B2})$$

and $Z_{0m} = \sqrt{\rho_0 C_0}$. Here Z_{0t} is defined in Eq. (8), with the diffusion branch chosen so that $\text{Re } \sqrt{s C_0 / \kappa_0} \geq 0$. The coefficient transfer matrix $K = Q_0^{-1} T Q_0$, partitioned into 2×2 blocks, gives

$$\begin{aligned} R^L &= -K_{22}^{-1} K_{21}, & R^R &= K_{12} K_{22}^{-1}, \\ T^R &= K_{22}^{-1}, & T^L &= K_{11} - K_{12} K_{22}^{-1} K_{21}. \end{aligned} \quad (\text{B3})$$

For unit-temperature thermal incidence, the physical stress transfer functions used in the Letter are related to the modal cross coefficients

by

$$\begin{aligned} H^L &= sZ_{0m} r_{\times}^L, & H^R &= -sZ_{0m} r_{\times}^R, \\ J^L &= -sZ_{0m} t_{\times}^L, & J^R &= sZ_{0m} t_{\times}^R. \end{aligned} \quad (\text{B4})$$

These signs give exactly the parity-adjusted ratios $\Lambda_{r,t}$ in Eq. (7). Note that reciprocity is satisfied with respect to the complete two-channel matrix:

$$\begin{aligned} C_0 R^{L,R} &= (C_0 R^{L,R})^T, \\ C_0 T^R &= (T^L)^T C_0, & C_0 &= \text{diag} \left(2sZ_{0m}, -\frac{2}{s\theta_R Z_{0t}} \right). \end{aligned} \quad (\text{B5})$$

Thus, it pairs opposite sides with exchanged mechanical/thermal channels and does not require the same thermal-to-mechanical transmission coefficient for opposite incidence directions.

For the local retrieval, permute to $\tilde{y} = (u, \theta, \sigma, -q)^T$ and set $P = (u, \theta)^T$, $\Psi = (\sigma, -q)^T$, $\Pi = (p, \theta_R \eta)^T$, and $G_s = \text{diag}(s, 1)$. Partition Eq. (3) as $\Psi = AP_{,x} + BG_s P$ and $\Pi = CP_{,x} + DG_s P$. If M_{ij} are the 2×2 blocks of $\tilde{M}_{\text{ret}} = l^{-1} \ln T$ in this ordering, comparison with the associated first-order system gives

$$\begin{aligned} A &= M_{12}^{-1}, & B &= -AM_{11}G_s^{-1}, \\ C &= s^{-1}M_{22}A, & D &= (s^{-1}M_{21} - CM_{11})G_s^{-1}. \end{aligned} \quad (\text{B6})$$

Reading these four blocks recovers the local scattering-equivalent specialization of Eq. (3). In the low-frequency calculations the logarithm was taken after diagonal similarity scaling and evaluated with the convergent $\ln(I + \Delta)$ series; re-exponentiation reproduced the exact heterogeneous T with maximum relative error 6.1×10^{-16} . This is an algebraic consistency check. The second BCH order also gives $\tilde{\gamma} = O(sl^2)$, establishing the order quoted after Eq. (6); its lengthy coefficient is not used in the plotted results.

Inverting the leading relation in Eq. (8) gives the two scattering estimates plotted in Fig. 2(b),

$$\tilde{\alpha}_{\text{scat}}^{(j)} = -\tilde{\beta} \tilde{\kappa} Z_{0t} \frac{1 - \Lambda_j}{1 + \Lambda_j}, \quad j = r, t. \quad (\text{B7})$$

All calculations use $\theta_R = 300$ K. Writing each material as $(C/\text{GPa}, \rho, \kappa, c/10^6, \beta/10^6)$ in SI units for the unscaled entries gives SiO_2 (73, 2200, 1.38, 1.65, 0.037), diamond (1050, 3520, 719, 1.78, 1.000), Cu (117.2, 8910, 391.1, 3.43, 1.983), and the decoupled host (130, 2330, 148, 1.66, 0). For the 25-nm $\text{SiO}_2|25\text{-nm diamond}$ slab, the static retrieved values entering Eq. (9) are $\tilde{\alpha}(0) = -1.62034 \times 10^{-3} \text{ Pa m K}^{-1}$, $\tilde{\beta}(0) = 9.95993 \times 10^4 \text{ Pa K}^{-1}$, and $\tilde{\kappa}(0) = 2.75471 \text{ W m}^{-1} \text{ K}^{-1}$.

For Fig. 3, the 99.9%-energy band is $4.01\text{--}6.44 \times 10^6 \text{ rad s}^{-1}$; for the selected 96-nm $\text{SiO}_2|24\text{-nm Cu}$ slab, $\max |k_m|L = 2.13 \times 10^{-4}$ and $\max |k_{\text{th}}|L = 0.333$. Independent layer-mode matching reproduces all four exact pulse histories with relative error below 1.2×10^{-13} . Removing the complete reciprocal $\tilde{\alpha}$ pair leaves 0.668% of the reflected and 0.676% of the transmitted orientation-odd L^2 norms, whereas ablating any other reciprocal pair changes either odd norm by less than 0.49%. This attribution is therefore specific to the scattering-equivalent representation of this subwavelength slab.

For Fig. 4, the exact 350-nm $\text{SiO}_2|1650\text{-nm Cu}$ slab has $\max |k_t|L \approx 30$. Its transmitted peaks are 0.0131 and 12.17 kPa K^{-1} for opposite incidence directions. Because this example is far outside the local thermal homogenization regime, no local- $\tilde{\alpha}$ attribution is made.